

Graduate Course

# B.A. (PROGRAMME) 1 YEAR

## ALGEBRA AND CALCULUS

**SM – 1**

**PART – A: (ALTEBRA)**

## VECTOR SPACES AND MATRICES

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## VECTOR SPACES

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### 1.1 Introduction

You are already familiar with several algebraic structures such as groups, rings, integral domains and fields. In this lesson we shall tell you about another equally important algebraic structures, namely, a *vector space*.

Let  $V$  be a non-empty set and let  $F$  be a field. Let us agree to call elements of  $V$  *vectors* and elements of  $F$  *scalars*.

A mapping from  $V \times V$  to  $V$  will be called addition in  $V$  and a mapping from  $F \times V$  to  $V$  will be called *multiplication by a scalar multiplication*,  $V$  is said to a vector space over  $F$  if addition and scalar multiplication satisfy certain properties. Of course, these conditions are to be chosen in such a manner that the resulting algebraic structure is rich enough to be useful. Before presenting the definition of a vector space, let us note that addition in  $V$  is denoted by the symbol '+', and scalar multiplication is denoted by juxtaposition, *i.e.*, if  $x \in V$ ,  $y \in V$ , and  $\alpha \in F$ , the  $x + y$  denoted the sum of  $x$  and  $y$ , and  $\alpha x$  denotes the scalar multiple of  $x$  by  $\alpha$ .

**Defintion 1.** *A non-empty set  $V$  is said to be a vector space over a field  $F$  with respect to addition and scalar multiplication if the following properties hold.*

**V 1** *Addition in  $V$  is associative, i.e.,*

$$x + (y + z) = (x + y) + z, \text{ for all } x, y, z, \in V$$

**V 2** *There exists of natural element for additon in  $V$ , i.e., there exists an element  $0 \in V$  such that*

$$x + 0 = 0 + x = x, \text{ for all } x \in V$$

**V 3** *Every element of  $V$  possesses a negaive (or additon inverse), i.e., for each  $x \in V$ , there exists an element  $y \in V$  such that*

$$x + y = y + x = 0.$$

**V 4** *Addition in  $V$  is commulative, i.e., for all elements  $x, y \in V$ ,*

$$x + y = y + x$$

**V 5** *Associativity of scalar multiplication, i.e.,*

$$\text{i.e., } \alpha(\beta x) = (\alpha \beta) x, \text{ for all } \alpha, \beta, \in F \text{ and } x \in V$$

**V 6** *Property of 1. For all  $x \in V$ ,*

$$1x = x, \text{ where } 1 \text{ is the multiplicative identity of } F.$$

**V 7** *Distributivity properties for all  $\alpha, \beta \in F$  and  $x, y \in V$ .*

$$(\alpha + \beta)x = \alpha x + \beta x$$

$$\alpha(x + y) = \alpha x + \alpha y$$

**Remarks 1.** The first of the two distributivity properties stated in V 7 above is generally called distributivity of *scalar multiplication over addition in  $F$* , and the second of the two distributivity properties is called *distributivity of scalar multiplication over addition in  $V$* .

2. We generally refer to properties V 1 – V 7 above by saying that  $(V, +)$  is a vector space over  $F$ . If the underlying field  $F$  is fixed, we simply say that  $(V, +)$  is a vector space, and do not make an explicit reference to  $F$ .

In case, the two vector space compositions are known, we denote a vectors space over a field'  $F$  by the symbol  $V(F)$ . If there is no chance of confusion about the underlying field, then we simply talk of 'the vector space  $V$ '.

3. You might have observed that the axioms V 1 to V 4 simply assert the  $V$  is an abelian group for the composition '+'. In view of the we can re-state the definition of a vector space as follows:

## 2. Definnition and Exampks of a Vector Space

**Defintion 2.** A triple  $(V, +)$  is said to be a vector space over a field  $F$  if  $(V, +)$  is an abelian group, and the following properties are satisfied :

$$\begin{aligned} \alpha(\beta x) &= (\alpha\beta)x, & \forall \alpha, \beta \in F \text{ and } \forall x \in V \\ 1x &= x, & \forall x, \beta \in V, \text{ where } 1 \text{ is the multiplicative identity of } F \\ (\alpha + \beta)x &= \alpha x + \beta x, & \forall \alpha, \beta \in F, \text{ and } \forall x, y \in V \\ \alpha(x + y) &= \alpha x + \alpha y, & \forall \alpha \in F \text{ and } \forall x, y \in V \end{aligned}$$

We shall now consider some examples of vector spaces.

**Example 1.** Let  $R$  be the set of number  $(R, +)$  is vector space over  $R$ . The addition is addition in  $R$  and scalar multiplication is simply multiplication of real numbers.

It is easy to verify that all the vector space axioms are verified. In fact, V1-4 are satisfied because  $R$  is an abelian group with respect to addition, V5 is nothing but the associative property of multiplication, V6 is the property of the multiplicative identity in  $(R, +)$  and the properties listed in V7 are nothing but the distributivity of multiplication over addition.

**Example 2.**  $(C, +, \cdot)$  is a vector space over  $C$

**Example 3.**  $(Q, +, \cdot)$  is a vector space over  $Q$ .

**Example 4.** Let  $F$  be *any* field.  $F$  is a vector space over itself for the usual compositions of addition and multiplication (to be called scalar multiplication) in  $F$ .

**Example 5.**  $C$  is a vector space over  $R$ , and  $R$  is a vector space over  $Q$ .

**Example 6.**  $R$  is not a vector space over  $C$ . Observe that if  $\alpha \in C$  and  $x \in R$ , the  $\alpha x$  is not in  $R$ . Therefore the multiplication composition in  $R$  fails to give rise to the scalar multiplication composition.

The examples considered above are in a way re-labelling of the field properties  $C, R$  or  $Q$ . We shall now consider some examples of a different type.

**Example 7.** Let  $V$  be the set of all vectors in a plane. You know that addition of two vectors is a vector, and that  $V$  is a group with respect to sum of vectors. Let us take addition of vectors as the first composition for the purpose of our example. Also, we know that if  $d$  be any vector and  $k$  be any real numbers, then  $k d$  is a vector. Let us take  $R$  as the underlying field and multiplication of vector by a scalar as the second vector space composition. It is easy to see that  $V$  is vector space over  $R$  for these two compositions.

**Example 8.** Let  $R^3$  be the set

$$\{(x_1, x_2, x_3) : x_1, x_2, x_3 \in R\}$$

and the addition and scalar multiplication  $R^3$  be defined as follows:

If  $x \in R^3$  and  $y \in R^3$  let

$$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3).$$

Also if  $x \in R^3$  and  $c \in R$ , let

$$cx = (cx_1, cx_2, cx_3).$$

It can be seen that  $R^3$  is a vector space over  $R$  for the two compositions—addition and scalar multiplication, as defined above.

We may note before passing on to the next example that the vector space being considered here is nothing but the space of the vectors (in space) with addition and scalar multiplication as the composition. This example is of special interest because it was in fact motivation for the present terminology of vector spaces.

The next three examples are a little abstract in nature but are quite important.

**Example 9.** Let  $R^n$  be the set of ordered  $n$ -tuples of real numbers, so that a typical element of  $R^n$  is  $(x_1, x_2, x_3, \dots, x_n)$ . We shall denote this element by  $x$  (printed as a bold-face better), and write

$$x = (x_1, x_2, \dots, x_n)$$

Let us take  $R$  to be the underlying field and define addition and scalar multiplication in  $R^n$  by setting

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n),$$

where

$$x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n)$$

and

$$\alpha x = (\alpha x_1, \alpha x_2, \dots, \alpha x_n), \forall \alpha \in R$$

Let us first of all see that addition scalar multiplication as defined above are meaningful in the sense that they define the two compositions that we need for making  $R^n$  vector space.

Since  $x_1, x_2, x_3, \dots, x_n$  and  $y_1, y_2, \dots, y_n$  are all real numbers, therefore  $x_1 + y_1, x_2 + y_2, \dots, x_n + y_n$  are all real numbers and therefore  $(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$  is an ordered  $n$ -tuple of real numbers and consequently it is in  $R^n$ . Again, since  $\alpha$  is a real number and  $(x_1, x_2, \dots, x_n)$  are also real numbers, therefore  $\alpha x_1, \alpha x_2, \dots, \alpha x_n$  are also real numbers, and consequently  $(\alpha x_1, \alpha x_2, \dots, \alpha x_n)$  is an  $n$ -tuple of real numbers and so is  $R^n$ .

Having defined addition and scalar multiplication in  $R^n$ , let us see in some detail that all the properties needed for  $R^n$  to be a vector space are actually satisfied.

1. Let  $x = (x_1, x_2, \dots, x_n)$ ,  $y = (y_1, y_2, \dots, y_n)$ ,  $z = (z_1, z_2, \dots, z_n)$  be any three elements of  $R^n$ .

$$\begin{aligned} \text{Then } (x + y) + z &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) + (z_1, z_2, \dots, z_n) \\ &= [(x_1 + y_1) + z_1, (x_2 + y_2) + z_2, \dots, (x_n + y_n) + z_n] \\ &= [x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), \dots, x_n + (y_n + z_n)] \\ &= (x_1, x_2, x_3, \dots, x_n) + (y_1 + z_1, y_2 + z_2, \dots, y_n + z_n) \\ &= x + (y + z). \end{aligned}$$

2. Let  $o = (0, 0, \dots, 0)$ . so that  $o \in R^n$  and  $x + 0 = o + x = x$  for all  $x \in R^n$

3. Let  $x = (x_1, x_2, \dots, x_n)$  be any element of  $R^n$ . If  $y = (-x_1, -x_2, \dots, -x_n)$  then  $y \in R^n$  and  $y$  is an element of  $R^n$  such that

$$x + y = y + x = 0$$

4. If  $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n)$  are any two elements of  $R^n$  then

$$\begin{aligned} x + y &= (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) \\ &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \\ &= (y_1 + x_1, y_2 + x_2, \dots, y_n + x_n) \\ &= (y_1, y_2, \dots, y_n) + (x_1, x_2, \dots, x_n) \\ &= y + x. \end{aligned}$$

5. If  $x = (x_1, x_2, \dots, x_n)$ , be any element of  $R^n$  and  $p, q$  be any real numbers, then

$$\begin{aligned} (pq)x &= (pq)(x_1, x_2, \dots, x_n) \\ &= [(pq)x_1, (pq)x_2, \dots, (pq)x_n] \\ &= [p(qx_1), p(qx_2), \dots, p(qx_n)] \\ &= p(qx_1, qx_2, \dots, qx_n) \\ &= p(qx) \end{aligned}$$

6. If  $x = (x_1, \dots, x_n)$  be any element of  $R^n$  and  $p, q$  be any real numbers, then

$$\begin{aligned} (p + q)x &= [(p + q)x_1, (p + q)x_2, \dots, (p + q)x_n] \\ &= (px_1, px_2, \dots, px_n) + (qx_1, qx_2, \dots, qx_n) \\ &= px + qx \end{aligned}$$

7. If  $x = (x_1, x_2, \dots, x_n)$  and  $y = (y_1, y_2, \dots, y_n)$  be any two elements of  $R^n$ , and  $p$  be any real number, then

$$\begin{aligned} p(x + y) &= p(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \\ &= [p(x_1 + y_1), p(x_2 + y_2), \dots, p(x_n + y_n)] \\ &= (px_1, px_2, \dots, px_n) + (py_1, py_2, \dots, py_n) \\ &= p(x_1, x_2, \dots, x_n) + p(y_1, y_2, \dots, y_n) \\ &= px + py. \end{aligned}$$

8. If  $x = (x_1, x_2, \dots, x_n)$  be any element of  $R^n$ , then

$$\begin{aligned} 1x &= 1(x_1, x_2, \dots, x_n) \\ &= (1 \cdot x_1, 1 \cdot x_2, \dots, 1 \cdot x_n) \\ &= (x_1, x_2, \dots, x_n) \\ &= x. \end{aligned}$$

From 1-8 above we find that  $R^n$  is a vector space over  $R$  with co-ordinatewise addition and co-ordinatewise scalar multiplication as the two vector space compositions.

The use of the word co-otherwise, is due to the fact that if  $x = (x_1, x_2, \dots, x_n)$  be any element of  $R^n$  then  $x_1, x_2, \dots, x_n$  are called the co-ordinates of  $x$ , and while defining  $x + y$ , we add the corresponding co-ordinates of  $x$  and  $y$ , and while defining  $cx$  we multiply the co-ordinates of  $x$  by  $c$ .

We may note that the space in example 9 is only a spacing case of the example 9 for  $n = 3$ .

**Example 10.** The set  $C^n$  of all ordered  $n$ -tuples of complex number is a vector space over  $C$  for co-ordinatewise addition and co-ordinatewise scalar multiplication as the two vector space compositions.

**Example 11.** Let  $F$  be any field. The set  $F^n$  of all ordered  $n$ -tuples of elements of  $F$  is a vector space over  $F$  with co-ordinatewise addition and co-ordinatewise scalar multiplication as the two vector space compositions.

**Example 12.** Let  $M_{mn}$  is a vector space over  $C$  with respect to matrices over  $C$ .  $M_{mn}$  is a vector space over  $C$  with respect to matrix addition and multiplication of a matrix by a scalar, for

1. The sum of two  $m \times n$  matices with complex entries is an  $m \times n$  matrix with complex entries.
2. Addition of matrices is associative.
3. The  $m \times n$  zero-matrix is a natural element for addition.
4. It  $A$  be an  $m \times n$  matrix with complex entries, then  $-A$  is also an  $m \times n$  with complex entries such that  $(-A) + A + (-A) = 0$
5. Addition of matrices is commutative.
6. If  $A \in M_{mn}$  and  $c$  be any complex number, then  $-A$  is also  $m \times n$  matrix with complex entries and so  $cA \in M_{mn}$
7. If  $p, q$  be any complex numbers and  $A, B$  be any two  $m \times n$  matrices with complex entries, then

$$\begin{aligned} p(q A) &= (pq) A, \\ (p + q) A &= pA + qA, \\ p(A + B) &= pA + pB, \\ 1 A &= A, \text{ for all } A \in M_{mn} \end{aligned}$$

**Example 13.** The set  $S$  of all matrices of the form  $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ , where  $a, b$  are any complex numbers, is a vector space over  $C$  with respect to matrix addition and multiplication by a scalar for the following reasons:

1. If  $A, B, \in S$ , the  $A + B \in S$ .

For, if 
$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix},$$

then 
$$A + B = \begin{pmatrix} a+c & b+d \\ -(b+d) & a+c \end{pmatrix} = \begin{pmatrix} p & q \\ -q & p \end{pmatrix}, \text{ where } p = a+c, q = b+d.$$

2. Matrix addition is associative.

3. The matrix  $O = \begin{pmatrix} o & o \\ o & o \end{pmatrix} \in S$ , and  $A + O = O + A = A$ , for all  $A \in S$ .

4. If  $A = \begin{pmatrix} p & q \\ -q & p \end{pmatrix} \in S$ , then the matrix  $B = \begin{pmatrix} -p & -q \\ q & -p \end{pmatrix} \in S$ , and is such that  $A + B = B + A = 0$ .

5. Addition of matrices is commutative.

6. If  $c$  be any complex number, and  $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in S$ , then  $cA$  is the matrix given by

$$cA = \begin{pmatrix} ca & cb \\ -cb & ca \end{pmatrix}.$$

It is obvious that  $cA \in S$ .

7. If  $p, q$  be any complex numbers, and  $A, B$  be any two matrices in  $S$ , say  $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ ,  $B = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$ ,

then

$$(i) \quad (pq)A = pq \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{bmatrix} (pq)a & (pq)b \\ (pq)(-b) & (pq)a \end{bmatrix} = \begin{bmatrix} p(qa) & p(qb) \\ p(-qb) & p(qa) \end{bmatrix} = p(qA).$$

$$(ii) \quad (p + q)A = pA + qA$$

$$(iii) \quad p(A + B) = pA + pB$$

$$(iv) \quad 1A = A.$$

**Example 14.** The set of all matrices of the form  $\begin{pmatrix} x & y \\ z & o \end{pmatrix}$ , where  $x, y, z, \in C$ , is a vector space

over  $C$  with respect to matrix addition and multiplication of a matrix by a scalar.

The verification of the vector space axioms is straight forward.

**Example 15.** The set  $S$  of all hermitian matrices of order  $n$  is a vector space over  $R$  with respect to matrix addition and multiplication of matrix by a scalar.

To verify that  $(S, +, \cdot)$  is a vector space over  $C$ , we proceed as follows:

1. Let  $A, B$  be two hermitian matrices of order  $n$ . Then  $A + B$  is a matrix of order  $n$ . It is hermitian because  $(A + B)^\theta = A^\theta + B^\theta = A + B$ , since  $A^\theta = A, B^\theta = B$ .
2. Addition of matrices is associative.
3. The  $n$ -rowed, zero matrix  $O$  is a matrix is a hermitian matrix such that  $A + 0 = 0 + A = A$
4. If  $A \in S$ , so that  $A^t = A$ , then  $(-A)^t = -A^t = -A$ , so that  $-A \in S$ , and  $A + (-A) = (-A) + A = 0$ .

5. Matrix addition is commutative

6. If  $C \in R$ , and  $A \in S$ , then  $(CA)^t = CA^t = CA$ , so that  $\bar{c}A \in S$ .

7. If  $p, q, \in R$  and  $A, B \in S$ , then

$$(i) \quad (p, q)A = p(qA)$$

$$(ii) \quad (p + q)A = pA + qA$$

$$(iii) \quad p(A + B) = pA + pB$$

$$(iv) \quad 1A = A$$

In view of the above properties it follows that  $(S, +, \cdot)$  is a vector space over  $R$ .



**Example 16.** The set  $S$  of all real symmetric matrices of order  $n$  is a vector space over  $\mathbb{R}$  with respect to matrix addition and multiplication of a matrix by a scalar.

In order to convince ourselves that  $(S, +, \cdot)$  is a vector space over  $\mathbb{R}$ , let us note the following:

1.  $A, B$  be  $n$ -rowed real symmetric matrices, then  $A^t = A, B^t = B$ . It follows that

$$(A + B)^t = A^t + B^t = A + B$$

2. Addition of matrices is associative, so that for all  $A, B, C \in S$ ,

$$A + (B + C) = (A + B) + C.$$

3. The  $n$ -rowed zero matrix  $0$  is a real symmetric matrix, and therefore it is in  $S$ .

Also, 
$$A + 0 = 0 + A = A$$

for all  $A$  in  $S$ .

4. If  $A \in S$  so that  $A^t = A$ , then  $(-A)^t = -A^t = -A$ , so that  $-A \in S$ . Also,  $A + (-A) = (-A) + A = 0$ .

5. Matrix addition is commutative.

6. If  $c \in \mathbb{R}$  and  $A \in S$ , then  $(cA)^t = cA^t = cA$ , so that  $cA \in S$ .

7. If  $p, q \in \mathbb{R}$ , and  $A, B \in S$ , then

(i)  $(p, q)A = p(qA)$

(ii)  $(p + q)A = pA + qA$

(iii)  $p(A + B) = pA + pB$

(iv)  $1A = A$

### 3. Some Direct Consequents of Vector Space Axioms

We shall now state and prove some elementary consequences of the vector space axioms. These will help us in dealing with vector in a convenient way in many situations.

**Theorem.** Let  $V$  be a vector over a field  $F$ . Then for all  $a \in F$  and  $x \in V$ ,

(i)  $\alpha_0 = 0,$

(ii)  $0x = 0,$

(iii)  $(-\alpha)x = -\alpha x$

(iv)  $(-\alpha)(-x) = \alpha x$

(v)  $\alpha x = 0$  if either  $\alpha = 0$  or  $x = 0$ .

**Proof.**

(i) 
$$\begin{aligned} \alpha_0 &= \alpha(0 + 0), \text{ by the property of } 0 \in V \\ &= \alpha 0 + \alpha 0, \text{ by distributivity of scalar multiplication over addition in } V \end{aligned}$$

Since  $\alpha 0 + \alpha 0 = \alpha 0 = \alpha 0 + 0$ , by the property of  $0$  in  $V$  by cancellation law in  $(V, +)$ , it follows that  $\alpha 0 = 0$

(ii) 
$$\begin{aligned} 0x &= (0 + 0)x, \text{ by the property of } 0 \text{ in } F \\ &= 0x + 0x, \text{ by distributivity of scalar multiplication over addition in } F. \end{aligned}$$

Since  $0x + 0x = 0x + 0$ , by the property of  $0$  in  $V$ , therefore by cancellation law in  $(V, +)$ , it follows that

$$0x = 0$$

$$\begin{aligned}
(iii) \quad 0 &= a + (-\alpha) && \Rightarrow 0x = (\alpha + (-\alpha))x \\
&&& \Rightarrow 0 = \alpha x + (-\alpha) x, \text{ since } \alpha x = 0.
\end{aligned}$$

Now  $\alpha x, (-\alpha)x$  are two elements of  $V$  such that  $\alpha x + (-\alpha)x = 0$ , therefore  $(-\alpha)x$  is the negative of  $\alpha x$ , i.e.,  $(-\alpha)x = -(\alpha x)$ .

$$\begin{aligned}
(iv) \text{ Now } & -\alpha(0) = -\alpha(x + (-x)) \\
\Rightarrow & 0 = -\alpha x + ((-\alpha)(-x)), \text{ since } -\alpha(0) = 0, \text{ by (i) above} \\
\Rightarrow & (-\alpha)(-x) \text{ is the negative of } -\alpha x \text{ in } V \\
\Rightarrow & (-\alpha)(-x) = -(-\alpha x) = \alpha x, \\
& \text{because } -\alpha x \in V \text{ and therefore negative of } -\alpha x \text{ in } V \text{ is simply } \alpha x.
\end{aligned}$$

(v) Let us first suppose that  $\alpha x = 0$ . If  $\alpha = 0$ , we are done. If  $\alpha \neq 0$ , then  $\alpha^{-1} \in F$ , because  $\alpha \in F$  and  $F$  is a field

$$\begin{aligned}
\text{Therefore } 0 &= \alpha^{-1} 0 \\
&= \alpha^{-1} (\alpha x), \text{ because } \alpha x = 0 \text{ by hypothesis} \\
&= (\alpha^{-1} \alpha)x, \text{ by associativity of scalar multiplication} \\
&= 1x \\
&= x.
\end{aligned}$$

Thus  $\alpha x = 0 \Rightarrow$  either  $\alpha = 0$  or  $x = 0$ . Conversely, let us assume that either  $\alpha = 0$  or  $x = 0$

In case  $\alpha = 0$ , by (i) above  $\alpha x = 0x = 0$ .

In case  $x = 0$ , by (i) above  $\alpha x = \alpha 0 = 0$ .

Thus in both cases we find that  $\alpha x = 0$ .

It is now obvious that  $\alpha x = 0$  if either  $x = 0$  or  $\alpha = 0$ .

## Exercise

1. Show that the set

$$C^2 = \{(x_1, x_2) : x_1 \in C, x_2 \in C\}$$

is a vector space over  $C$  with respect to co-ordinateswise addition and scalar multiplication.

2. Show that the set of all  $2 \times 2$  matrices over  $C$  is a vector space over  $C$  with respect to matrix addition and multiplication of a matrix by a scalar.

3. Let  $V = \{(a_1, a_2, a_3, a_4) : a_1, a_2, a_3, a_4 \text{ are integers}\}$

Is  $V$  a vector space over  $R$  with respect to co-ordinatewise addition and scalar multiplication? Justify your answer.

4. Let  $V = \{(x_1, x_2, x_3) : (x_1, x_2, x_3, \text{ are complex numbers, and } x_1, x_2, = 0)\}$

Is  $V$  a vector space over  $C$  with respect to co-ordinateswise addition and scalar multiplication? Justify your answer.

5. Show that the set of all matrices of the form  $\begin{pmatrix} x & o \\ o & y \end{pmatrix}$ , where  $y \in C$  is a vector space over  $C$  with respect to matrix addition and multiplication of a matrix by a scalar.

6. Show that the set of all matrices of the form  $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ , where  $a, b \in C$  is vector space over  $C$  with respect to matrix addition and multiplication of matrix by a scalar.

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## MATRICES

### (BASIC CONCEPTS)

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### 1. Introduction

You are already familiar with addition and multiplication of matrices. We shall now talk about some important types of matrices such as symmetric and skew-symmetric matrices, hermitian and skew-hermitian matrices etc., elementary operations on a matrix inverse of a matrix, rank of a matrix, and characteristic equation of a matrix. In the end we shall apply some of these concepts to solutions of systems of linear equations. However, before we do so, we shall briefly recapitulate the main facts about addition and multiplication of matrices.

### 2. Definition of a Matrix

Let  $S$  be any set. A set of  $mn$  elements arranged in a rectangular array of  $m$  rows and  $n$  column as

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

is called an  $m \times n$  (“ $m$  by  $n$ ”) matrix over  $S$ . A matrix may be represented by the symbols  $\| a_{ij} \|$ ,  $[a_{ij}]$ ,  $[a_{ij}]$  or by a single letter such as  $A$ . The  $a_{ij}$ ’s in a matrix are called the element of the matrix. The indices  $i$  and  $j$  of an element indicate respectively the row and the column in which the elements  $a_{ij}$  is located.

Since we shall be dealing only with matrices over the set of complex number therefore, we shall use the word “matrix” so as to mean “matrix over  $C$ ” throughout, unless we state to the contrary.

The  $1 \times n$  matrices are called *row vectors* and the  $m \times 1$  matrices are called *column vectors*. The  $m \times n$  matrix whose elements are 0 is called the *null matrix* (or zero matrix) of the type  $m \times n$ . It is usually denoted by  $O_{m \times n}$  or simply by  $O$  if there is no possibility of confusion.

If the number of rows and the number of columns of a matrix are equal (say each equal to  $n$ ) the matrix is said to be a *square matrix* of order  $n$  or an  $n$ -row square matrix. The elements  $a_{11} a_{22}, \dots a_{nn}$  of a square matrix  $A$  are said to constitute the *main diagonal* of  $A$ . A square matrix in which all the off diagonal elements are zero is called a *diagonal matrix*. Thus an  $n$ -rowed square matrix  $[a_{ij}]$  is a diagonal matrix if  $a_{ij} = 0$  whenever  $i \neq j$ . An  $n$ -rowed diagonal matrix is often written as

$$\text{dia. } [a_{11}, a_{22}, \dots, a_{nn}]$$

A diagonal matrix in which all the diagonal elements are equal is called a *scalar matrix*. In other words, an  $n$ -rowed square matrix  $[a_{ij}]$  is a scalar matrix if for some number  $k$ .

$$a_{ij} = \begin{cases} k, & \text{when } i = j, \\ 0, & \text{when } i \neq j. \end{cases}$$

A scalar matrix in which each diagonal element is unity, is called a unit matrix. Thus, an  $n$ -rowed square matrix  $[a_{ij}]$  is called a unit matrix if

$$a_{ij} = \begin{cases} 1, & \text{whenever } i = j, \\ 0, & \text{whenever } i \neq j. \end{cases}$$

The  $n$ -rowed unit matrix is usually denoted by  $I_n$  (or simply by  $I$  if there is no possibility of confusion).

The matrix of elements which remain after deleting any number of rows and columns of a matrix  $A$  is called a sub matrix of  $A$ .

**Illustrations :**

1.  $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$  is the  $3 \times 4$  null matrix.

2.  $\begin{pmatrix} 3 & 1 & 2 \\ 5 & 4 & 7 \\ -1 & 2 & 8 \end{pmatrix}$  is a 3-rowed square matrix. 3, 4, 8 constitute the main diagonal of this matrix.

3.  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{pmatrix}$  is a 3-rowed diagonal matrix.

4.  $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$  is a 3-rowed scalar matrix.

5.  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  is the 3-rowed unit matrix. We denote it by  $I_3$ .

6. The matrix  $\begin{pmatrix} 3 & 4 \\ 6 & -5 \end{pmatrix}$  is submatrix of  $\begin{pmatrix} 1 & 8 & 7 \\ -2 & 3 & 4 \\ 1 & 6 & -5 \end{pmatrix}$  because it can be obtained from the latter by deleting the first row and the first column.

**3. Equality of Matrices**

Two matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are said to be equal if (i) they are comparable, i.e., the number of rows in  $B$  is the same as the number of rows in  $A$ , and the number of columns in  $B$  is the same as the number of columns in  $A$ ; (ii)  $a_{ij} = b_{ij}$  for every pair of subscripts  $i$  and  $j$ . Thus, for example,

the matrices  $\begin{pmatrix} 3 & 7 \\ 8 & 9 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 5 & 4 \\ 3 & 6 & 2 \end{pmatrix}$  are not comparable ; the matrices  $\begin{pmatrix} -1 & 2 & 3 \\ 3 & 1 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 4 & 3 & 6 \\ 1 & 8 & 9 \end{pmatrix}$

are comparable but not equal; the matrices  $\begin{pmatrix} 2 & 4 & 7 \\ 6 & 3 & -1 \end{pmatrix}$  and  $\begin{pmatrix} \sqrt{4} & 4 & 7 \\ 2.3 & \sqrt{9} & -1 \end{pmatrix}$  are equal.

From the definition of equality of matrices, it can be easily verified that if  $A$ ,  $B$ , and  $C$  be any matrices, then

- (i)  $A = A$  (reflexivity)
- (ii)  $A = B \Rightarrow B = A$  (symmetry)
- (iii) if  $A = B$  and  $B = C$ , then  $A = C$  (transitivity)

The above statements (i)—(iii) can be summed up by saying that *the relation of equality in the set of all matrices is an equivalence relation.*

#### 4. Addition of Matrices

If  $A = [a_{ij}]$ , and  $B = [b_{ij}]$  be two matrices of the same type, say  $m \times n$ , their sum is the  $m \times n$  matrix  $C = [c_{ij}]$ , where  $c_{ij} = a_{ij} + b_{ij}$  for every pair of subscripts  $i$  and  $j$ . In other words,

$$\text{If } A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

$$\text{then } A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{pmatrix}$$

**Illustrations.** If  $A = \begin{pmatrix} -2 & 1 & 3 \\ 4 & 2 & -1 \end{pmatrix}$  and  $B = \begin{pmatrix} 4 & 2 & 5 \\ 3 & 0 & 6 \end{pmatrix}$

$$\text{then } A + B = \begin{pmatrix} -2 + 4 & 1 + 2 & 3 + 5 \\ 4 + 3 & 2 + 0 & -1 + 6 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 8 \\ 7 & 2 & 5 \end{pmatrix}$$

#### Properties of matrix addition

Addition of matrices, has the following properties:

- (i) *Addition of matrices is associative.* That is, if  $A$ ,  $B$ , and  $C$  be matrices of the same type, then  $A + (B + C) = (A + B) + C$
- (ii) *Addition of matrices is commutative.* That is, if  $A$  and  $B$  be matrices of the same type, then  $A + B = B + A$ .
- (iii) *Property of zero matrix.* If  $A$  be an  $m \times n$  matrix and  $0$  denotes the  $m \times n$  zero matrix, then  $A + 0 = 0 + A = A$ .
- (iv) *Negative of a matrix.* If  $A$  be an  $m \times n$  matrix, there exists an  $m \times n$  matrix  $B$ , called the negative of the matrix  $A$ , such that  $A + B = B + A = 0$ .

#### 5. Multiplication of a Matrix by a Scalar

If  $A = [a_{ij}]$  be an  $m \times n$  matrix, and  $k$  be any complex number, then  $kA$  is defined to be the  $m \times n$  matrix whose  $(i, j)^{\text{th}}$  element is  $k a_{ij}$ . The matrix  $kA$  is called the scalar multiple of  $A$  by  $k$ . The following properties of scalar multiplication are worth noting :

- (i) If  $A$  and  $B$  are comparable matrices, and  $k$  is any complex number, then

$$k(A + B) = kA + kB.$$

- (ii) If  $A$  be any matrix, and  $k$  and  $l$  be any two complex numbers, then

$$(k + l) A = kA + lA.$$

(iii) If  $A$  be any matrix, and  $k$  and  $l$  be any two complex numbers, then

$$k(lA) = (kl)A.$$

(iv) For every matrix  $A$ ,

$$1A = A$$

## 6. Multiplication of Matrices

**Definition 1.** Let  $A = [a_{ij}]$  and  $B = [b_{ij}]$  be  $m \times n$  and  $n \times p$  matrices respectively. The  $m \times p$  matrix  $[c_{ij}]$ , where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{k=1}^n a_{ik}b_{kj},$$

is called the product of the matrices  $A$  and  $B$  and is denoted by  $AB$ .

The above definition expresses two facts:

- (i) We can talk of the product  $AB$  if two matrices of and only if the number of columns of  $A$  is equal to the number of rows of  $B$ . In the case this condition is satisfied, we say that  $A$  and  $B$  are conformable to multiplication.
- (ii) If  $A$  and  $B$  are conformable to multiplication, the  $(i, j)^{\text{th}}$  element of the matrix  $AB$  is obtained by multiplying the elements of the  $i^{\text{th}}$  row of  $A$  by the corresponding elements of the  $j^{\text{th}}$  column of  $B$  and adding the products. The sum so obtained is the desired  $(ij)^{\text{th}}$  elements of  $AB$ .

### Properties of matrix multiplication

The following are some of the important properties of matrix multiplication :

- (i) *Matrix multiplication is associative.* That is, if  $A$ ,  $B$ , and  $C$  be of suitable sizes for the products  $A(BC)$  and  $(A + B)C$  to exist, then  $A(BC) = (AB)C$ .
- (ii) *Matrix multiplication is not commutative.* That is, given two matrices  $A$  and  $B$ ,  $AB = BA$  is not always true. It is important to note here that for pair of matrices  $A$  and  $B$  several different possibilities arise.
  - (a) Neither of the products  $AB$  and  $BA$  exists;
  - (b) only one of the products  $AB$  and  $BA$  exist and the other one does not exist;
  - (c) both  $AB$  as well as  $BA$  exist but they are of different type;
  - (d) both  $AB$  as well as  $BA$  exist and are of the same type, but are not equal
  - (e)  $AB = BA$ .

All the above possibilities do exist for certain pairs of matrices.

The important thing to note is that the phrase 'matrix multiplication is not commutative' means that  $AB$  is not always equal to  $BA$ . It does not exclude the possibility of  $AB$  and  $BA$  being equal in some cases.

(iii) *Multiplication of matrices is distributive with respect to addition, i.e.,*

$$A(B + C) = AB + AC$$

and  $(B + C)D = BD + CD$

where  $A$ ,  $B$ ,  $C$  and  $D$  are of the suitable sizes for the above relations to be meaningful.

(iv) *Multiplication by the unit matrix.* If  $A$  be any  $m \times n$  matrix then  $I_m A = A = A I_n$

## 7. Positive Integral Powers of Sources Matrix

If  $A$  be an  $n$ -rowed square matrix, and  $n$  be a positive integer, then  $A^n$  is defined by setting  $A^1 = A$ ,  $A^{k+1} = A^k A$ . By the principle of finite induction this defines  $A^n$  for all positive integer  $A_n$ .

If  $A$  be an  $n$ -rowed square matrix and  $p$  and  $q$  be positive integers, it can be easily shown that

$$A^p \cdot A^q = A^{p+q}, (A^p)^q = A^{pq}.$$

## 8. Transpose of a Matrix

Consider the matrices

$$A = \begin{pmatrix} 3 & 1 & -4 \\ 6 & 0 & 7 \end{pmatrix}, B = \begin{pmatrix} 3 & 6 \\ 1 & 0 \\ -4 & 7 \end{pmatrix}.$$

The matrix  $A$  is a  $2 \times 3$  matrix, and the matrix  $B$  is a  $3 \times 2$  matrix. Also, the first column of  $B$  is the same as the first row of  $A$ , and the second column of  $B$  is the same as the second row of  $A$ . In other words,  $B$  is the matrix obtained from  $A$  by writing the row of  $A$  as columns. We say that the matrix  $B$  is the transpose of  $A$ .

**Defintion 2.** If  $A = [a_{ij}]$  be an  $m \times n$  matrix, then the  $n \times m$  matrix  $B = [b_{ij}]$ , such that  $b_{ij} = a_{ji}$  is called the transpose of  $A$  and is deonted by  $A^t$ .

From the above defintion we find that

- (i) the transpose of an  $m \times n$  matrix is an  $n \times m$  matrix ;
- (ii) the  $(i, j)^{\text{th}}$  element of  $A^t$  is the  $(j, i)^{\text{th}}$  element of  $A$ .

**Example 1.** Let  $A = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 4 & 7 \end{pmatrix}$  and  $B = \begin{pmatrix} -1 & 3 & 5 \\ 6 & 2 & 1 \end{pmatrix}$

Compute  $A^t$ ,  $(A^t)^t$ ,  $B^t$ ,  $(A + B)^t$ ,  $A^t + B^t$ ,  $(3A)^t$ ,  $3A^t$ .

**Solution.**  $A^t = \begin{pmatrix} 3 & 0 \\ -1 & 4 \\ 2 & 7 \end{pmatrix}$   $(A^t)^t = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 4 & 7 \end{pmatrix}$

$$B^t = \begin{pmatrix} -1 & 6 \\ 3 & 2 \\ 5 & 1 \end{pmatrix}.$$

$$A + B = \begin{pmatrix} 2 & 2 & 7 \\ 6 & 6 & 8 \end{pmatrix}, (A + B)^t = \begin{pmatrix} 2 & 6 \\ 2 & 6 \\ 7 & 8 \end{pmatrix}, A^t + B^t = \begin{pmatrix} 2 & 6 \\ 2 & 6 \\ 7 & 8 \end{pmatrix}.$$

$$3A = \begin{pmatrix} 9 & -3 & 6 \\ 0 & 12 & 21 \end{pmatrix}, (3A)^t = \begin{pmatrix} 9 & 0 \\ -3 & 12 \\ 6 & 21 \end{pmatrix}, 3A^t = \begin{pmatrix} 9 & 0 \\ -3 & 12 \\ 6 & 21 \end{pmatrix}.$$

**Remark.** In the above example we find that  $(A^t)^t = A$ ,  $(A + B)^t = A^t + B^t$ , and  $(3A)^t = 3A^t$ . These results are only special cases of the following theorem :

**Theorem 1.** If  $A^t$  and  $B^t$  are transposes of  $A$  and  $B$  respectively, then

(i)  $(A^t)^t = A$

(ii)  $C(A + B)^t = A^t + B^t$ , if  $A$  and  $B$  are comparable.

(iii)  $kA^t = (kA)^t$ ,  $k$  being any complex number

**Proof.** (i) Let  $A = [a_{ij}]$  be an  $m \times n$  matrix. Then  $A^t$  is an  $n \times m$  matrix and  $(A^t)^t$  is an  $m \times n$  matrix. The matrices  $(A^t)^t$  and  $A$  are, therefore, comparable.

Also,

$$\begin{aligned} & (i, j)^{\text{th}} \text{ element of } (A^t)^t \\ &= (j, i)^{\text{th}} \text{ element of } A^t \\ &= (i, j)^{\text{th}} \text{ element of } A \end{aligned}$$

Since the matrices  $(A^t)^t$  and  $A$  are comparable and their  $(i, j)^{\text{th}}$  elements are equal, therefore,  $(A^t)^t = A$ .

(ii) Let  $A = [a_{ij}]$  and  $B = [b_{ij}]$  be  $m \times n$  matrices. Since  $A$  and  $B$  are both  $m \times n$  matrices, therefore  $A + B$  exists and is an  $m \times n$  matrix. Consequently  $(A + B)^t$  is an  $n \times m$  matrix, therefore

Again,  $A^t$  and  $B^t$  are both  $n \times m$  matrices, so that  $A^t + B^t$  also exists and is an  $n \times m$  matrix. The matrices  $(A + B)^t$  and  $A^t + B^t$  are both of the type  $n \times m$ , and are therefore comparable.

Also,  $(i, j)^{\text{th}}$  element  $(A + B)^t = (j, i)^{\text{th}}$  element of  $A + B$

$$\begin{aligned} &= a_{ji} + b_{ji} \\ &= (j, i)^{\text{th}} \text{ element of } A^t + (i, j)^{\text{th}} \text{ element } B^t \\ &= (j, i)^{\text{th}} \text{ element of } (A + B)^t \end{aligned}$$

Thus the matrices  $(A + B)^t$  and  $A^t + B^t$  are comparable, and their corresponding elements are equal.

Hence  $(A + B)^t = A^t + B^t$ .

(iii) Let  $A = [a_{ij}]$  be an  $m \times n$  matrix.  $kA$  is an  $m \times n$  matrix and therefore  $(kA)^t$  is an  $n \times m$  matrix. Also  $A^t$  being an  $n \times m$  matrix,  $kA^t$  is an  $n \times m$  matrix. The matrices  $(kA)^t$  and  $kA^t$  are both of type  $n \times m$ , and are, therefore, comparable. Also  $(i, j)^{\text{th}}$  element of  $(kA)^t = (j, i)^{\text{th}}$  element of  $kA$

$$\begin{aligned} &= ka_{ji} \\ &= k[(i, j)^{\text{th}} \text{ elements of } A^t] \end{aligned}$$

Since the matrices  $(kA)^t$  and  $kA^t$  are comparable and their  $(i, j)^{\text{th}}$  elements are equal, therefore

$$(kA)^t = kA^t.$$

Remark. If  $A^t = B$ , the  $B^t = (A^t)^t = A$ , i.e., if  $B$  is the transpose of  $A$ , then  $A$  is the transpose of  $B$ .

**Example 2.** If  $A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 0 & -3 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 3 & 4 \\ -2 & 1 & -1 \\ 0 & -4 & 2 \end{pmatrix}$ ,

Compute  $(AB)^t$  and  $B^tA^t$ .

**Solution.**

$$\begin{aligned} AB &= \begin{pmatrix} 3 & -1 & 2 \\ 1 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 4 \\ -2 & 1 & -1 \\ 0 & -4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 0 & 17 \\ 1 & 15 & -2 \end{pmatrix}, \end{aligned}$$



so that 
$$(AB)^t = \begin{pmatrix} 5 & 1 \\ 0 & 15 \\ 17 & -2 \end{pmatrix}$$

Also, 
$$B^t = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 1 & -4 \\ 4 & -1 & 2 \end{pmatrix}, A^t = \begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 2 & -3 \end{pmatrix}.$$

Therefore 
$$B^t A^t = \begin{pmatrix} 5 & 1 \\ 0 & 15 \\ 17 & -2 \end{pmatrix}.$$

**Remark.** In the above example  $(AB)^t = B^t A^t$ . This is of course, only a particular case of the general result which we state and prove in the following theorem.

**Theorem 2.** *If  $A$  and  $B$  be matrices conformable to multiplication, then*

$$(AB)^t = B^t A^t$$

**Proof.** Let  $A = [a_{ij}]$  and  $B[b_{ij}]$  be  $m \times n$  and  $n \times p$  matrices particularly.

The  $A^t = [c_{ij}]$ , where  $c_{ij} = a_{ji}$ , is an  $n \times m$  matrix.

$B^t = [d_{ij}]$ , where  $d_{ij} = b_{ji}$ , is an  $p \times n$  matrix. The matrices  $(AB)^t$  and  $B^t A^t$  are both of type of  $p \times m$ , and are therefore comparable.

Also  $(i, j)^{\text{th}}$  elements of  $(AB)^t$

$$= (j, i)^{\text{th}} \text{ element of } (AB)$$

$$= \sum_{k=1}^n a_{jk} b_{ki}$$

$$= \sum_{k=1}^n c_{kj} d_{ik}$$

$$= \sum_{k=1}^n d_{ik} c_{kj}$$

$$= (i, j)^{\text{th}} \text{ element of } B^t A^t$$

Since the matrices  $(AB)^t$  and  $B^t A^t$  are of the same type, and their  $(ij)^{\text{th}}$  elements are equal, therefore,  $(AB)^t = B^t A^t$

## 9. Symmetric and Skew-Symmetric Matrices

Consider the matrices

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 4 & -1 \\ 2 & -1 & 5 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & -4 \\ -3 & 4 & 0 \end{pmatrix}$$

In matrix  $A$ ,  $(1, 2)^{\text{th}}$  element is equal to  $(2, 1)^{\text{th}}$  element,  $(1, 3)^{\text{th}}$  element is equal to  $(3, 1)^{\text{th}}$  element, and  $(2, 3)^{\text{th}}$  element is equal to  $(3, 2)^{\text{th}}$  element. Because of these properties we say that the matrix  $A$  is symmetric.

In matrix  $B$ ,  $(2, 1)^{\text{th}}$  element is the negative of  $(1, 2)^{\text{th}}$  element,  $(3, 1)^{\text{th}}$  element is the negative of the  $(1, 3)^{\text{th}}$  element,  $(3, 2)^{\text{th}}$  element is the negative of the  $(2, 3)^{\text{th}}$  element, and  $(1, 1)^{\text{th}}$  element,  $(2, 2)^{\text{th}}$  element, and  $(3, 3)^{\text{th}}$  element are own negatives, (*i.e.*, they are all zero). Because of these properties we say that the matrix  $B$  is skew-symmetric.

Symmetric and skew-symmetric matrices play useful (an important) roles in the theory of matrices.

**Definition 3.** A square matrix  $A = [a_{ij}]$  is said to be symmetric if  $a_{ij} = a_{ji}$  for all  $i$  and  $j$ .

**Illustrations 1.** The matrices

$$\begin{pmatrix} 4 & 1-i & 2 \\ 1-i & 3 & 7 \\ 2 & 7 & i \end{pmatrix} \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ are all symmetric.}$$

2. The matrices

$$\begin{pmatrix} 0 & i & 1+i \\ -i & 0 & -3 \\ -1-i & 3 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 3 & 6-i \\ -3 & 0 & -4 \\ -6+i & 4 & 0 \end{pmatrix} \text{ are both skew-symmetric}$$

3. The matrices

$$\begin{pmatrix} 1 & i & 1+i \\ -i & 2 & 6i \\ -1-i & 6 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & i \\ 2 & -i & 0 \end{pmatrix} \text{ are neither symmetric nor skew-symmetric.}$$

In the following theorem we state and prove some basic facts about symmetric and skew-symmetric matrices.

**Theorem 3.**

(i) A necessary and sufficient condition for a matrix  $A$  to be symmetric is that  $A^t = A$ .

(ii) A necessary and sufficient condition for a matrix  $A$  to be skew-symmetric is that  $A^t = -A$ .

(iii) The diagonal elements of a skew-symmetric matrix are all zero.

**Proof.** (i) **Necessity.** Let  $A = [a_{ij}]$  be a symmetric matrix. Since  $A$  is symmetric, it must be square matrix, say of order  $n$ ,  $A^t$  is then also of order  $n$ , so that  $A^t$  and  $A$  are comparable. Also,  $(i, j)^{\text{th}}$  element of  $A^t = a_{ji} = a_{ij} = (i, j)^{\text{th}}$  elements of  $A$ .

Therefore  $A^t = A$ .

**Sufficiently.** Let  $A = [a_{ij}]$  be an  $m \times n$  matrix such that  $A^t = A$ . Since  $A$  is an  $m \times n$  matrix., therefore  $A^t$  is an  $m \times n$  matrix. Since  $A^t$  and  $A$  are equal matrices, they are comparable, so that  $n = m$ , and consequently  $A$  is a square matrix. Also, as given,  $(i, j)^{\text{th}}$  element of  $A^t = (i, j)^{\text{th}}$ , which gives  $a_{ji} = a_{ij}$ .

Since  $A$  a square matrix such that  $a_{ij} = a_{ji}$  for all  $i$  and  $j$ , therefore  $A$  is symmetric.

(ii) **Necessity.** Let  $A = [a_{ij}]$  be a skew-symmetric matrix. Since  $A$  is skew-symmetric, it must be a square matrix, say of order  $n$ .  $A^t$  is then also of order  $n$ , so that  $A^t$  and  $A$  are comparable. Also,  $(i, j)^{\text{th}}$  element of  $A^t = a_{ji} = -a_{ij} = (i, j)^{\text{th}}$  element of  $-A$  therefore  $A^t = -A$ .

**Sufficiently.** Let  $A = [a_{ij}]$  be an  $m \times n$  matrix such that  $A^t = -A$ . Since  $A$  is  $m \times n$  matrix, therefore  $A^t$  is an  $n \times m$  matrix and  $-A$  is an  $m \times n$  matrix. Since the matrices  $A^t$  and  $-A$  are equal, they are comparable, so that  $n = m$ , and consequently  $A$  is a square matrix. Also  $(i, j)^{\text{th}}$  element of  $A^t = -(i, j)^{\text{th}}$  element of  $A$ , which gives  $a_{ji} = -a_{ij}$ .

We shall now state and prove a theorem which assures us that every square matrix can be expressed uniquely as a sum of a symmetric and skew-symmetric matrix.

**Theorem 4.** *Every square matrix can be expressed uniquely as the sum of a symmetric and a skew-symmetric matrix.*

**Proof.** Let  $A$  be an  $n$ -rowed square matrix.

$$\text{Let } A = X + Y,$$

where  $X$  is an  $n$ -rowed symmetric, and  $Y$  is an  $n$ -rowed skew-symmetric matrix. Taking the transpose of both sides of (1), we have

$$A^t = (X + Y)^t = X^t = Y^t = X - Y \quad \dots(2)$$

since  $X$  is symmetric and  $Y$  is skew-symmetric so that  $X^t = X$  and  $Y^t = -Y$

From (1) and (2), we get

$$X = \frac{1}{2}(A + A^t), \quad \dots(3)$$

$$Y = \frac{1}{2}(A - A^t) \quad \dots(4)$$

We have shown that if  $A$  is expressible as the sum of a symmetric matrix  $X$  and a skew-symmetric matrix  $Y$ , then  $X$  and  $Y$  must be given by (3) and (4). This establishes the uniqueness part. To demonstrate the existence of a symmetric matrix,  $X$  and a skew-symmetric matrix  $Y$  such that  $A = X + Y$ , we have only to see that if we write.

$$X = \frac{1}{2}(A + A^t), Y = \frac{1}{2}(A - A^t),$$

then

$$\begin{aligned} X^t &= \left[ \frac{1}{2}(A + A^t) \right]^t, \\ &= \frac{1}{2} [A^t + (A^t)^t], \\ &= \frac{1}{2}(A^t + A), \\ &= X, \\ Y^t &= \left[ \frac{1}{2}(A - A^t) \right]^t \\ &= \frac{1}{2} [A^t - A^{tt}] \\ &= \frac{1}{2}(A^t - A) \\ &= -Y, \end{aligned}$$

so that  $X$  is an  $n \times n$  symmetric matrix and  $Y$  is an  $n \times n$  skew-symmetric matrix. Furthermore,  $X + Y = A$ , which completes the proof.

**Example 3.** Express the matrix

$$A = \begin{pmatrix} 2 & 6 & 5 \\ 3 & 1 & 4 \\ 9 & -1 & 7 \end{pmatrix}$$

as the sum of symmetric and a skew-symmetric matrix.

**Solution.** Let

$$\begin{pmatrix} 2 & 6 & 5 \\ 3 & 1 & 4 \\ 9 & -1 & 7 \end{pmatrix} = X + Y,$$

where  $X$  is a 3-rowed symmetric matrix and  $Y$  is a 3-rowed skew-symmetric matrix.

Taking transposes of both sides (1), and using the facts that  $(X + Y)^t = X^t + Y^t$ ,  $X^t = X$ ,  $Y^t = -Y$ , we have

$$\begin{pmatrix} 2 & 3 & 6 \\ 6 & 1 & -1 \\ 5 & 4 & 7 \end{pmatrix} = X - Y.$$

From (1) and (3) we find that

$$2X = \begin{pmatrix} 4 & 9 & 14 \\ 9 & 2 & 3 \\ 14 & 3 & 14 \end{pmatrix}, \quad 2Y = \begin{pmatrix} 0 & 3 & -4 \\ -3 & 0 & 5 \\ 4 & -5 & 0 \end{pmatrix},$$

so that

$$X = \begin{pmatrix} 2 & \frac{9}{2} & 7 \\ \frac{9}{2} & 1 & \frac{3}{2} \\ 7 & \frac{3}{2} & 7 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & \frac{3}{2} & -2 \\ -\frac{3}{2} & 0 & \frac{5}{2} \\ 2 & -\frac{5}{2} & 0 \end{pmatrix}$$

**Verification.** Since  $X^t = X$ ,  $Y^t = -Y$ , therefore that matrix  $X$  is symmetric and the matrix  $Y$  is skew-symmetric. Also, by actual addition we find that  $X + Y = A$ .

## 10. Hermitian and Skew-Hermitian Matrice

Let  $A = [a_{ij}]$  be an  $m \times n$  matrix. The  $m \times n$  matrix  $B = [b_{ij}]$  such that  $b_{ij} = \bar{a}_{ij}$  is called the conjugate of  $A$  and is denoted by  $\bar{A}$ . The transpose of  $A$  is called the tranposed conjugate or transjugate of  $A$  and is enoted by  $A^\theta$ .

For example

If 
$$A = \begin{pmatrix} 3+2i & 4i \\ -7 & 3-4i \end{pmatrix},$$

then 
$$\bar{A} = \begin{pmatrix} 3-2i & -4i \\ -7 & 3+4i \end{pmatrix},$$

$$A^\theta = \begin{pmatrix} 3-2i & -7 \\ -4i & 3+4i \end{pmatrix}$$

It can be easily the seen that :

(i) if  $A$  be any matrix, then  $(A^\theta)^\theta = A$ .

(ii) if and  $B$  be two matrices conformable to addition, then

$$(A + B)^\theta = A^\theta + B^\theta$$

(iii) if  $A$  be any matrix and  $k$  be any complex number, then

$$(kA)^\theta = \bar{k}A^\theta.$$

(iv) if  $A$  and  $B$  be two matrix conformable to multiplication, then

$$(AB)^\theta = B^\theta A^\theta.$$

We have the proofs of (i), (ii) to the reader and prove only (iii).

Let  $A = [a_{ij}]$  be an  $m \times n$  matrix and let  $B = [b_{ij}]$  be an  $n \times p$  matrix.

The matrices  $(AB)^\theta$ ,  $B^\theta$ ,  $A^\theta$  are respectively of types  $p \times m$ ,  $p \times n$ , and  $n \times m$ , so that  $B^\theta A^\theta$  is a  $p \times m$  matrix.

Since  $(AB)^\theta$  and  $B^\theta A^\theta$  are both the type  $p \times m$ , therefore they are comparable. Also  $(ij)^\text{th}$  element of

$$(AB)^\theta = (j, i)^\text{th} \text{ element of } (\overline{AB}) = \left( \overline{\sum_{k=1}^n a_{jk} b_{ki}} \right) = \sum_{k=1}^n \bar{a}_{jk} \bar{b}_{ki}$$

$$\begin{aligned} \text{Also, } (i, j)^\text{th} \text{ element of } B^\theta A^\theta &= \sum_{k=1}^n [(ik)^\text{th} \text{ element of } B^\theta \times (kj)^\text{th} \text{ element of } A^\theta] \\ &= \sum_{k=1}^n [(ki)^\text{th} \text{ element of } \bar{B} \times (jk)^\text{th} \text{ element of } \bar{A}] \\ &= \sum_{k=1}^n \bar{b}_{ki} \bar{a}_{jk} \\ &= \sum_{k=1}^n \bar{a}_{kj} \bar{b}_{ki}, \end{aligned}$$

which is the same as the  $(i, j)^\text{th}$  element of  $(AB)^\theta$ .

Since (i)  $(AB)^\theta$  and  $B^\theta A^\theta$  are matrices of the same type, and (ii)  $(ij)^\text{th}$  elements of  $(A, B)^\theta$  and  $B^\theta A^\theta$  are equal for all  $i$  and  $j$ , therefore  $(AB)^\theta = B^\theta A^\theta$ .

**Defintion 4.** A square matrix  $A = [a_{ij}]$  is said to be Hermitian if  $\bar{a}_{ij} = a_{ji}$  for all  $i$  and  $j$ . A square matrix  $A [a_{ij}]$  is said to be skew-hermitina if  $\bar{a}_{ij} = -a_{ji}$  for all  $i$  and  $j$ .

**Illustration 1.** The matrices

$$\begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3+4i \\ 3-4i & -4 \end{pmatrix} \text{ are Hermitian.}$$

2. The matrices

$$\begin{pmatrix} 2i & 1+3i \\ -1+3i & 5i \end{pmatrix}, \begin{pmatrix} 1 & 5-7i \\ -5-7i & 3i \end{pmatrix} \text{ are skew-hermitian.}$$

3. The matrix  $\begin{pmatrix} i & 4+i \\ 4-i & 1 \end{pmatrix}$  is neither Hermitian nor skew-hermitian.

In the following theorem we prove the important facts about hermitian and skew-hermitian matrices.

**Theorem 5.** Let  $A$  be an  $n$ -rowed square matrix.

- (i) A necessary and sufficient condition for  $A$  to be hermitian is that  $A^\theta = A$ .
- (ii) A necessary and sufficient condition for  $A$  to be skew-hermitian is that  $A^\theta = -A$ .
- (iii) The diagonal elements of a hermitian matrix are all real, and the diagonal elements of a skew hermitian matrix are either pure imaginaries of zero.

**Proof.** (i) First let us assume that  $A$  is hermitian. Since  $A$  is an  $n$ -rowed square matrix, therefore  $A^\theta (= (\bar{A})^t)$  is also an  $n$ -rowed square matrix. Consequently  $A$  and  $A^\theta$  are matrices of the same type. Also,  $(i, j)$ <sup>th</sup> element of  $A^\theta$   $(j, i)$ <sup>th</sup> element of  $\bar{A} = \bar{a}_{ji} = a_{ij} = (i, j)$ <sup>th</sup> element of  $A$ . Hence  $A^\theta = \bar{A}$ . Conversely, let us suppose that  $A^\theta = A$ .  $(i, j)$ <sup>th</sup> element of  $A^\theta = \bar{a}_{ji}$ . Since  $A^\theta = A$ , therefore  $\bar{a}_{ji} = a_{ij}$ , i.e.,  $\bar{a}_{ij} = (\bar{a}_{ji}) = a_{ji}$ , and consequently  $A$  is hermitian.

(ii) First let us assume that  $A$  is skew-hermitian. Since  $A$  is an  $n$ -rowed square matrix, therefore,  $A^\theta$  and  $-A$  are both  $n$ -rowed square matrices, and as such they are comparable. Also  $(i, j)$ <sup>th</sup> element of  $A^\theta = (j, i)$ <sup>th</sup> element of  $A = \bar{a}_{ji}$ , and  $(i, j)$ <sup>th</sup> element of  $-A = -a_{ij}$ . Since  $A$  is skew-hermitian, therefore  $\bar{a}_{ji} = -a_{ij}$ . Hence  $(i, j)$ <sup>th</sup> element of  $A^\theta = (i, j)$ <sup>th</sup> element of  $-A$ .

Since  $A^\theta$  and  $-A$  are comparable, and their  $(i, j)$ <sup>th</sup> elements are equal for all  $i$  and  $j$ , therefore  $A^\theta = -A$ .

Conversely, let us suppose that  $A^\theta = -A$  then  $(i, j)$ <sup>th</sup> elements of  $A^\theta$  and  $-A$  must be equal for all  $i$  and  $j$ . This gives  $\bar{a}_{ji} = -a_{ij}$ . Taking conjugates of both sides, we have  $a_{ji} = -\bar{a}_{ij}$ , i.e.,  $\bar{a}_{ij} = a_{ji}$ , so that  $A$  is skew-hermitian.

(iii) Let  $A$  be a hermitian matrix. then  $\bar{a}_{ij} = a_{ji}$  for all  $i$  and  $j$ . For a diagonal elements  $j = i$ , which gives  $\bar{a}_{ij} = a_{ji}$  for all  $i$ , i.e.,  $a_{ji}$  thus real. Thus the diagonal elements of  $A$  are all real.

If  $A$  is skew-hermitian,  $\bar{a}_{ij} = -a_{ji}$  for all  $i, j$ . For  $j = i$  this gives  $\bar{a}_{ij} = -a_{ji}$ . If  $a_{ij} = p + iq$ , then  $a_{ji} = p - iq$  and so we have  $(p - iq) = -(p + iq)$  i.e.,  $2p = 0$  or  $p = 0$ . The real part of  $a_{ij}$  is, therefore, zero and consequently  $a_{ji}$  is either pure imaginary (if  $q \neq 0$ ).

We shall now state and prove a decomposition theorem which tells us that every square matrix can be uniquely decomposed as the sum of the hermitian and a skew-hermitian matrix.

**Theorem 6.** Every square matrix can be express uniquely in the form  $X + iY$ , where  $X$  and  $Y$  are hermitian matrices.

**Proof.** Let  $A$  be an  $n$ -rowed square matrix.

$$\begin{aligned} \text{Let } A &= (X + iY)^\theta \\ &= X^\theta + (iY)^\theta \\ &= X^\theta - iY^\theta \end{aligned}$$

$$= X - iY,$$

Since  $X^\theta = X$  and  $Y^\theta = Y$ .

From (1) and (2), we have

$$X = \frac{1}{2}(A + A^\theta), Y = \frac{1}{2i}(A - A^\theta)$$

We have thus shown that if  $A$  is expressible as  $X + iY$ , where  $X$  and  $Y$ , are hermitian, the  $XZ$  and  $Y$  must be given by (3). Thus establishes the uniqueness part.

To demonstrate the existence hermitian matrices  $X$  and  $Y$  such that  $A = X + iY$ , we have only to see that if we write

$$X = \frac{1}{2}(A + A^\theta), Y = \frac{1}{2i}(A - A^\theta)$$

then 
$$X^\theta = \left[ \frac{1}{2}(A + A^\theta) \right]^\theta$$

$$= \frac{1}{2}(A + A^{\theta\theta})$$

$$= \frac{1}{2}(A^\theta + A), \text{ since } A^{\theta\theta} = A.$$

$$= X,$$

$$Y^\theta = \left[ \frac{1}{2i}(A - A^\theta) \right]^\theta$$

$$= -\frac{1}{2i}(A^\theta - A^{\theta\theta})$$

$$= -\frac{1}{2i}(A^\theta - A), \text{ since } A^{\theta\theta} = A.$$

$$= Y,$$

so that  $X$  and  $Y$  are hermitian.

Also,  $X + iY = A$ , which completes the proof.

**Example 4.** Express the matrix.

$$A = \begin{pmatrix} 1 & 7 & 1-i \\ 1+i & i & 2 \\ -3 & 1 & 4+2i \end{pmatrix}$$

in the form  $X + iY$ , where  $X$  and  $Y$  are hermitian matrices.

**Solution.** We know that every square matrix  $A$  can be uniquely expressed in the form  $X + iY$ , where  $X$  and  $Y$  are hermitian matrices given by

$$X = \frac{1}{2}(A + A^\theta), Y = \frac{1}{2i}(A - A^\theta)$$

Here 
$$A^\theta = \begin{pmatrix} 1 & 1-i & -3 \\ 7 & -i & 1 \\ 1+i & 2 & 4-2i \end{pmatrix}$$

Therefore 
$$X^\theta = \frac{1}{2}(A + A^\theta) = \begin{pmatrix} 1 & 4-\frac{1}{2}i & -1-\frac{1}{2}i \\ 4+\frac{1}{2}i & 0 & \frac{3}{2} \\ -1+\frac{1}{2}i & \frac{3}{2} & 4 \end{pmatrix}$$

$$Y = \frac{1}{2i}(A - A^\theta) = \begin{pmatrix} 0 & \frac{1}{2}-3i & -\frac{1}{2}-2i \\ \frac{1}{2}+3i & 1 & -\frac{1}{2}i \\ -\frac{1}{2}+2i & \frac{1}{2}i & 2 \end{pmatrix}$$

**Verification.** Since  $X^\theta = X$  and  $Y^\theta = Y$ , the matrices  $X$  and  $Y$  are hermitian. Also  $X + iY = A$

### Exercise

1. For each of the following matrices  $A$ , verify that  $A^t = A$  :

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & -4 \\ -1 & -4 & 3 \end{pmatrix}, \begin{pmatrix} -1 & 6 & -7 \\ 6 & 3 & 8 \\ -7 & 8 & -5 \end{pmatrix}, \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$$

2. For each of the following matrices  $A$ , verify that  $A^t = -A$ .

$$\begin{pmatrix} 0 & i & 1 \\ -i & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 1+i \\ -3 & 0 & -4 \\ -1-i & 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -3 & 2i \\ 3 & 0 & 4 \\ -2i & -4 & 0 \end{pmatrix}$$

3. If  $A = \frac{1}{3} \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}$ , verify that  $A A^t = A^t A = A = I_3$ .

4. If  $A$  be any square matrix, verify that the matrix  $A + A^t$  is symmetric and the matrix  $A - A^t$  is skew-symmetric.
5. If  $A$  be any square matrix, prove that  $A + A^\theta$ ,  $AA^\theta$ ,  $A^\theta A$  are the hermitian and  $A - A^\theta$  is skew-hermitian.

6. Express the matrix  $\begin{pmatrix} 3 & 1 & -7 \\ 2 & 4 & 8 \\ 6 & -1 & 2 \end{pmatrix}$  as  $X + Y$  where  $X$  is symmetric and  $Y$  is skew-symmetric.

7. Find hermitian matrices  $A$  and  $B$  such that

$$A + iB = \begin{pmatrix} 1 & 1+i & 3 \\ 4 & 2 & 1-i \\ -3 & 2+3i & i \end{pmatrix}$$



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## ELEMENTARY OPERATIONS ON A MATRIX AND INVERSE OF A MATRIX

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### 1. Introduction

In this lesson we shall study two important concepts—namely elementary operations on a matrix, and inverse of a matrix. Both are devices for producing new matrices out of a given matrix. However, there is one difference. Elementary operations can be applied to a matrix of any type but the process of finding the inverse can be applied only to some special types of matrices. Furthermore, while it is possible to produce as many new matrices as we like from a given matrix by means of elementary operations, the operation of finding the inverse, if applicable, can produce only one matrix from a given matrix.

Both the types of operations that we are going to study have one thing in common. Both can be used for solving a system of linear equations. There is one important connection between the two types of operations. Elementary operations can be used to find the inverse of a matrix—and that is one reason for studying them together in this lesson.

### 2. Elementary Operations

Consider the matrices

$$S = \begin{pmatrix} -2 & 3 & 4 \\ 5 & 0 & -6 \end{pmatrix}, A = \begin{pmatrix} 5 & 0 & -6 \\ -2 & 3 & 4 \end{pmatrix}, B = \begin{pmatrix} -6 & 9 & 12 \\ 5 & 0 & -6 \end{pmatrix}, C = \begin{pmatrix} -2 & 3 & 4 \\ 1 & 6 & 2 \end{pmatrix},$$

$$D = \begin{pmatrix} 4 & 3 & -2 \\ -6 & 0 & 5 \end{pmatrix}, E = \begin{pmatrix} -2 & -6 & 4 \\ 5 & 0 & -6 \end{pmatrix}, F = \begin{pmatrix} -2 & 3 & 16 \\ 5 & 0 & -6 \end{pmatrix}.$$

The matrices  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$  and  $F$  are related to  $S$  in as much as :

$A$  can be obtained from  $S$  by interchanging the first and second rows.

$B$  can be obtained from  $S$  by multiplying the first row of  $S$  by 3.

$C$  can be obtained from  $S$  by adding 2 times the first row to the second row.

$D$  can be obtained from  $S$  by interchanging the first and third columns.

$E$  can be obtained from  $S$  by multiplying the second column of  $S$  by  $-2$ .

$F$  can be obtained from  $S$  by adding 4 times the second column of  $S$  to the third column.

The operations described above are only examples of operations known as elementary operations.

**Definition 1.** *An elementary operation is an operation of the following types:*

**Type I.** *Interchange of two rows of columns*

**Type II.** *Multiplication of a row or a column by a non-zero number*

**Type III.** *Addition of a multiple of one row or column to another row or column.*

*An elementary operation is called a row operation or a column operation according as it applies to rows or columns.*

It can be easily seen that if a matrix  $B$  is obtained from a matrix  $A$  by an E-operation then the matrix  $A$  can be obtained from  $B$  by an E-operation of the same type.

It is convenient (and usual) to use the following notation of E-operations :

We shall denote the interchange of  $i^{\text{th}}$  and  $j^{\text{th}}$  rows (columns) by  $R_i \leftrightarrow R_j$  ( $C_i \leftrightarrow C_j$ ); multiplication of  $i^{\text{th}}$  row (column) by a number  $k \neq 0$  by  $R_i \rightarrow kR_j$  ( $C_i \rightarrow + kC_i$ ); and addition of  $k$  times the  $j^{\text{th}}$  row (column) to the  $i^{\text{th}}$  row (column) by  $R_i \rightarrow R_i + k R_j$  ( $C_i \rightarrow C_i + kC_j$ ).

If a matrix  $B$  is obtained from a matrix  $A$  by a finite chain of E-operations, we say that  $A$  is equivalent to  $B$  and write it as  $A \sim B$ .

**Elementary Matrices.** A matrix obtained from a unit matrix by a single E-operation is called an elementary matrix of an E-matrix. For example, the matrices.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

are the E-matrices obtained from  $I_3$  by the E-operations  $R_1 \leftrightarrow R_2$  (or  $C_1 \leftrightarrow C_2$ ),  $R_2 \rightarrow 3R_2$  or ( $C_2 \rightarrow 3C_2$ ) and  $R_3 \rightarrow R_3 + 3R_1$  (or  $C_1 \rightarrow C_1 + 3C_3$ ) respectively.

**Remark.** It can be easily see that the operations  $R_i \leftrightarrow R_j$  and  $C_i \leftrightarrow C_j$  have the same effect on  $I_n$ ;  $R_i \rightarrow kR_i$  and  $C_i \rightarrow kC_i$  have the same effect on  $I_n$ ;  $R_i \rightarrow R_i + kR_j$  and  $C_j \rightarrow C_j + kC_i$  have the same effect on  $I_n$ . It an E-matrix is obtained from  $I_n$  by an E-operation  $\tau$ , we say that it is the E-matrix corresponding to the operation  $\tau$ .

*Effect of an elementary operation on the product of two matrices.*

Before we consider the effect of an E-operation on the product of two matrices, let us consider the following example.

**Example 1.** Let  $A = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 0 & -4 \\ -3 & 5 & 6 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 2 \\ -1 & 4 \\ 3 & -5 \end{pmatrix}$ .

Let  $C$  and  $D$  be the matrices obtained from  $A$  and  $AB$  respectively by the E-row operation  $R_1 \leftrightarrow R_2$ . Compute  $C$ ,  $D$  and  $CB$  and show that  $D = CB$ .

**Solution.**

$$AB = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 0 & -4 \\ -3 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 4 \\ 3 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 20 \\ -11 & 22 \\ 10 & -16 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & -4 \\ 3 & 1 & -2 \\ -3 & 5 & 6 \end{pmatrix},$$

$$D = \begin{pmatrix} -11 & 22 \\ -4 & 20 \\ 10 & -16 \end{pmatrix},$$

$$\begin{aligned}
CB &= \begin{pmatrix} 1 & 0 & -4 \\ 3 & 1 & -2 \\ -3 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 4 \\ 3 & -5 \end{pmatrix} \\
&= \begin{pmatrix} -11 & 22 \\ -4 & 20 \\ 10 & -16 \end{pmatrix} \\
&= D.
\end{aligned}$$

In the above example we find that the E-row operation  $R_1 \leftrightarrow R_2$  on the product  $AB$  is equivalent to the same E-operation on the pre-factor  $A$ . in other words.

Whether we apply the E-row operation  $R_1 \leftrightarrow R_2$  to the matrix  $A$  and then post-multiply the resulting matrix by  $B$ , or first multiply the matrices  $A$  and  $B$ , and then apply the E-row operation  $R_1 \leftrightarrow R_2$  to the product we get the same result.

The above example suggests the following theorem :

**Theorem 1.** *An elementary row operation on the product of two matrices is equivalent to the same elementary row operation on the pre-factor.*

**Proof.** Let  $A = [a_{ij}]$  and  $B = [b_{ij}]$  be  $m \times n$  and  $n \times p$  matrices respectively. We shall show that if  $A^*$  be the matrix obtained from  $A$  by an E-row operation, and  $(AB)^*$  be the matrix obtained from  $AB$  by the same E-row operations, then  $A^* B = (AB)^*$ .

We shall consider the three types of E-rows operations one by one and prove the result in each case.

**Type I.** Consider the E-row operation  $R_i \leftrightarrow R_j$

Let  $A^*$  and  $(AB)^*$  be the matrices obtained from  $A$  and  $B$  respectively by the E-row operation  $R_i \leftrightarrow R_j$  since  $A$  is of type  $m \times n$ , therefore  $A^*$  is also of type  $m \times n$ , and consequently  $A^* B$  is of type  $m \times p$ . Also since  $AB$  is of type  $m \times p$ ,  $(AB)^*$  is also of type  $m \times p$ . The matrices  $A^* B$  and  $(AB)^*$  are, therefore of the same type.

The matrices  $A^*$  and  $A$  differ from each other in the  $i^{\text{th}}$  and  $j^{\text{th}}$  rows only, therefore, it follows that  $A^* B$  and  $AB$  differ from each other in the  $i^{\text{th}}$  and  $j^{\text{th}}$  row only. Also  $(AB)^*$  and  $AB$  differ from each other in the  $i^{\text{th}}$  and  $j^{\text{th}}$  row only.

Consequently, the matrices  $A^* B$  and  $(AB)^*$  can differ at the most in  $i^{\text{th}}$  and  $j^{\text{th}}$  rows only. It follows that in order to prove the equality of  $A^* B$  and  $(AB)^*$  it is enough to show that the  $i^{\text{th}}$  (as well as  $j^{\text{th}}$ ) row of the matrices  $A^* B$  and  $(AB)^*$  are identical.

Now  $(i, k)^{\text{th}}$  element of  $A^* B$

$$\begin{aligned}
&= \sum_{r=1}^n a_{jr} b_{rk} \\
&= (j, k)^{\text{th}} \text{ element of } AB \\
&= (i, k)^{\text{th}} \text{ element of } (AB)^*;
\end{aligned}$$

showing that the  $i^{\text{th}}$  rows of  $A^* B$  and  $(AB)^*$  are identical.

Similarly the  $j^{\text{th}}$  rows of  $(AB)^*$  and  $A^* B$  are identical.

Hence  $A^* B = (AB)^*$

**Type II.** Consider the E-row operations  $R_i \leftrightarrow kR_i$  ( $k \neq 0$ )

Let  $A^*$  and  $(AB)^*$  be the matrices obtained from  $A$  and  $AB$  respectively by the E-row operation  $R_i \leftrightarrow kR_i$  ( $k \neq 0$ ).

As in the case of the Type I, (i) the matrices  $A * B$  and  $(AB)^*$  are of the same type (ii)  $A * B$  and  $(AB)^*$  can differ in the  $i^{\text{th}}$  row only at the most.

$$\begin{aligned}
 &= \sum_{p=1}^n \{(i, p)^{\text{th}} \text{ element of } A^* \cdot (p \cdot l)^{\text{th}} \text{ element of } B\} \\
 &= \sum_{p=1}^n ka_{ip}b_{pl} \\
 &= k \sum_{p=1}^n a_{ip}b_{pl} \\
 &= k \cdot \{(i, l)^{\text{th}} \text{ element of } AB\}. \\
 &= (i, l)^{\text{th}} \text{ element of } (AB)^*,
 \end{aligned}$$

so that the  $i^{\text{th}}$  rows of  $A * B$  and  $(AB)^*$  are identical.

Hence  $A * B = (AB)^*$

**Type III.** Let  $A^*$  and  $(AB)^*$  be the matrices obtained from  $A$  to  $AB$  respectively by the E-row operation  $R_i \rightarrow R_i + kR_j$ . As in the case of Type I, (i) the matrices  $A * B$  and  $(AB)^*$  are of the same type.

(ii)  $A * B$  and  $(AB)^*$  can differ in the  $i^{\text{th}}$  row at the most. Therefore in order to complete the proof it is enough to show that the  $i^{\text{th}}$  rows of  $A * B$  and  $(AB)^*$  are identical.

Now  $(i, j)^{\text{th}}$  element  $A * B$

$$\begin{aligned}
 &= \sum_{r=1}^n \{(i, r)^{\text{th}} \text{ element of } A^* \cdot (r \cdot l)^{\text{th}} \text{ element of } B\} \\
 &= \sum_{r=1}^n (a_{ir} + ka_{jr})b_{rl} \\
 &= \sum_{r=1}^n a_{ir}b_{rl} + k \sum_{r=1}^n a_{jr}b_{rl} \\
 &= (i, l)^{\text{th}} \text{ element of } AB + k \text{ times } = (j, l)^{\text{th}} \text{ element of } AB \\
 &= (i, l)^{\text{ss}} \text{ element of } (AB)^*,
 \end{aligned}$$

showing that the  $i^{\text{th}}$  rows of  $A * B$  and  $(AB)^*$  are identical.

Hence  $A * B = (AB)^*$

From the above we find that if  $A^*$  and  $(AB)^*$  be the matrices obtained from  $A$  and  $AB$  respectively by the same E-row operation of any one of the three types, then  $A * B = (AB)^*$ .

A theorem similar to the above holds for E-column operations.

**Theorem 2.** *An elementary column operation on the product of two matrices is equivalent to the same column operation on the post-factor.*

**Proof.** Given a matrix  $M$  and an E-column operation  $c$ , let us denote by  $c(M)$  the matrix obtained for  $M$  by the E-column operation  $c$ . Also, let us denote by  $c^*(M)$  the matrix obtained from  $M$  by the corresponding E-row operation.

Let  $A$  and  $B$  be two matrices of types  $m \times n$  and  $n \times p$  respectively. Denoting by  $c(A)$  and  $c(AB)$  the matrices obtained from  $A$  and  $AB$  respectively by the E-column operation  $c$ , we have

$$\begin{aligned} c(AB) &= [c^*(AB)]^t, \\ &= [c^*(B^t A^t)]^t, \\ &= [c^*(B^t) A^t]^t \\ &= (A^t)^t [c^*(B^t)]^t \\ &= Ac(B). \end{aligned}$$

This proves the theorem.

**Remark.** Instead of deducing the theorem 2 from theorem 1, we can prove it by imitating the proof of theorem 1.

As direct consequences of the above theorems we can prove the following important theorem.

**Theorem 3.** *Every elementary row (resp. column) operation on a matrix is equivalent to pre-multiplication (resp. post-multiplication) by the elementary matrix corresponding to that theorem.*

**Proof.** Let  $A = [a_{ij}]$  be an  $m \times n$  matrix. With the same notation as in the proof of theorem 2, we have

$$A^* = (I_m A)^* = I_m^* A,$$

from which we find that  $A^*$  can be obtained from  $A$  by pre-multiplying  $A$  by the elementary matrix  $I_m^*$ .

Again, writing

$$\begin{aligned} A &= AI_n, \text{ we have} \\ c(A) &= c(AI_n) \\ &= Ac(I_n), \end{aligned}$$

From which we find that  $c(A)$  can be obtained from  $A$  by post-multiplying  $A$  by the elementary matrix  $c(I_n)$ . Hence the theorem.

**Example 2.** Reduce the matrix.

$$A = \begin{pmatrix} -1 & 2 & 1 & 8 \\ 3 & 1 & -1 & 0 \\ 4 & 2 & 1 & 7 \end{pmatrix}$$

to triangular form by E-row operations.

**Solution.**

**Step 1.** By suitable E-row operations on  $A$  we shall reduce  $A$  to a matrix in which all elements in the first column except the first element are zero. We can do this by the E-row operations,  $\rightarrow R_2 + 3R_1$  and  $R_3 \rightarrow R_3 + 4R_1$ . These two operations yield

$$A \sim \begin{pmatrix} -1 & 2 & 1 & 8 \\ 0 & 7 & -2 & 24 \\ 0 & 10 & 5 & 39 \end{pmatrix} = B, \text{ say}$$

**Step 2.** By suitable E-row operations, we shall reduce B to a matrix in which all the elements in the second column except the first two are zero. Since we would like that in this process the elements of the first column remain unaltered, therefore we apply the E-row operations  $R_3 \rightarrow R_3 - \frac{10}{7}R_2$  to B. This will reduce the (3, 2)<sup>th</sup> element to zero and will not alter any element of the first column.

We thus get

$$B \sim \begin{pmatrix} -1 & 2 & 1 & 8 \\ 0 & 7 & -2 & 24 \\ 0 & 0 & \frac{55}{7} & \frac{33}{7} \end{pmatrix} = C, \text{ say.}$$

The matrix C is a triangular matrix. We thus find that the E-row operations  $R_2 \rightarrow R_2 + 3R_1$ ,  $R_3 \rightarrow R_3 - 4R_1$ ,  $R_3 \rightarrow R_3 - \frac{10}{7}R_2$  reduce A to the triangular matrix C.

**Remarks.** 1. If a matrix has more than three rows, then we shall have several steps which would be similar to step 2 of the above example.

2. If one of the elements in the first column is either 1 or  $-1$ , it is *convenient but not essential* to bring it to the (1, 1)<sup>th</sup> place as the initial step in reducing a matrix to triangular form.

3. If every element in the first column of the given matrix happens to be zero then step 1 of the above example is not required and we have to start with step 2.

4. If the (1, 1)<sup>th</sup> element is zero, but there is at least one element in the first column different from zero, we apply a suitable E-row operation and bring this non-zero element to (1, 1)<sup>th</sup> place. For example, in order to reduce the matrix

$$\begin{pmatrix} 0 & 3 & -1 \\ -2 & -4 & -5 \\ 3 & 1 & 9 \end{pmatrix}$$

to triangular form we apply the E-row operation  $R_1 \leftrightarrow R_2$  so as to obtain the matrix.

$$\begin{pmatrix} -2 & -4 & -5 \\ 0 & 3 & -1 \\ 3 & 1 & 9 \end{pmatrix}$$

in which the (1, 1)<sup>th</sup> element is different from zero. We can now effect triangular reduction of this matrix in the same way as the in the above example.

5. In the illustration in remark 4 above, we could as well have applied the E-row operation  $R_1 \leftrightarrow R_3$  to the given matrix to obtain a matrix in which the (1, 1)<sup>th</sup> element is different from zero.

6. Triangular reduction of a matrix is not unique. In fact, if we apply the E-row operation  $R_3 \rightarrow 7R_3$  on the matrix C in the above example, then

We get the matrix

$$\begin{pmatrix} -1 & 2 & 1 & 8 \\ 0 & 7 & -2 & 24 \\ 0 & 0 & 55 & 33 \end{pmatrix}$$

which is also a triangular matrix.

**Theorem 4.** Every matrix can be reduced to a triangular form by elementary row operations.

**Solution.** We shall prove the theorem by inductive on the number of rows.

Let  $A = [a_{ij}]$  be an  $m \times n$  matrix.

The theorem is trivially true when  $m = 1$ , for every matrix having only one row is a triangular matrix.

Let us assume that the theorem holds for all matrices having  $(m - 1)$  rows, i.e., every matrix having  $(m - 1)$  rows can be reduced to triangular form by E-row operations. We shall show that  $A$  can be reduced to triangular form by E-row operations.

Three different cases arise according as (i)  $a_{11} \neq 0$ , (ii)  $a_{11} = 0$  but  $a_{i1} \neq 0$  for some  $i$ , (iii)  $a_{i1} = 0$  for all  $i$ . We shall consider these cases one by one.

**Case (i).** Let  $a_{11} \neq 0$ . The E-row operation  $R_1 \rightarrow a_{11}^{-1} R_1$  reduces  $A$  to an  $m \times n$  matrix  $B = [b_{ij}]$  in which  $b_{11} = 1$ . The E-row operations  $R_f \rightarrow R_f - b_{f1} R_1$  ( $f = 2, \dots, m$ ) reduce  $B$  to an  $m \times n$  matrix  $C$  in which  $C_{f1} = 0$  whenever  $f > 1$ . The matrix  $C$  is of the form

$$\begin{pmatrix} 1 & c_{12} & c_{13} & \dots & c_{1n} \\ 0 & c_{22} & c_{23} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & c_{m2} & c_{m3} & \dots & c_{mn} \end{pmatrix}$$

By our hypothesis, the  $(m - 1)$  rowed matrix

$$\begin{pmatrix} c_{22} & c_{23} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{m2} & c_{m3} & \dots & c_{mn} \end{pmatrix}$$

can be reduced to triangular form by E-row operations. If we apply the corresponding E-row operations to  $C$ , it will be reduced to triangular form.

**Case (ii).** Let  $a_{11} = 0$ , but  $a_{i1} \neq 0$  for some  $f$  such that  $1 \leq f \leq m$ . By applying the E-row operation  $R_1 \rightarrow R_f$ , to  $A$  we get a matrix  $D = [d_{ij}]$  in which  $d_{11} = a_{f1} \neq 0$ .

We can now proceed as in case (i) and reduce  $D$  to triangular form by E-row operations.

**Case (iii).** If  $a_{i1} = 0$  for all  $i$  such that  $1 \leq i \leq m$ , (i.e., all the elements in the first column are zero), then  $A$  is of the form

$$\begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

By hypothesis we can reduce the  $(m - 1)$  - rowed matrix

$$\begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

to triangular form by E-row operations. When we apply the corresponding E-row operations to  $A$ , it will be reduced to triangular form.

From the above we find that in all the three cases the matrix  $A$  can be reduced to triangular form by E-row operations.

The proof is now complete by the principle of finite induction.

### Exercise 1

1. Apply the elementary operation  $R_2 \leftrightarrow R_3$  to the matrix

$$\begin{pmatrix} 1 & -1 & 3 \\ 4 & 2 & -6 \\ 5 & 8 & 9 \end{pmatrix}$$

2. Apply the elementary operation  $C_2 \rightarrow 2C_2$  to the matrix

$$\begin{pmatrix} -1 & 3 & 7 & 6 \\ 5 & -1 & 4 & -2 \end{pmatrix}$$

3. Write down the elementary matrix obtained by applying  $R_3 \rightarrow R_3 - 4R_1$  to  $I_3$ .
4. Reduce the matrix

$$\begin{pmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 2 & 0 \\ 2 & 1 & -1 & 5 \end{pmatrix}$$

to triangular form by applying E-row operations

5. Reduce the matrix

$$\begin{pmatrix} 1 & -1 & -1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{pmatrix}$$

to  $I_3$  by E-row operations.

6. Verify that the E-row equation  $R_1 \rightarrow R_1 - R_3$  on the matrix  $AB$  is equivalent to the same E-row operation on  $A$ , where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -3 & 1 & 4 \\ 0 & 1 & 3 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 & -5 \\ 1 & -2 & 6 \\ 3 & 1 & 1 \end{pmatrix}.$$

### 3. Inverse of a Matrix

Consider the matrices

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & -4 \\ -4 & -9 & 5 \end{pmatrix}, B = \begin{pmatrix} 4 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{pmatrix}.$$



It can be easily seen that

$$AB = BA = I_3.$$

Because of this relationship between  $A$  and  $B$  we say that the matrix  $B$  is an inverse of the matrix  $A$ . In fact, we have the following definition.

**Definition 2.** If  $A = [a_{ij}]$  be an  $n$ -rowed square matrix, then a square matrix  $B = [b_{ij}]$  is said to be an inverse of  $A$  if  $AB = BA = I_n$ .

Since the relation  $AB = BA$  remains unaltered when we interchange  $A$  and  $B$ , it follows that if a matrix  $B$  is an inverse of a matrix,  $A$  then  $A$  must be the inverse of  $B$ . Furthermore, it is obvious from the definition of an inverse of a matrix that if a matrix  $A$  has an inverse, then it must be a square matrix. In fact, if  $A$  be an  $m \times n$  matrix and  $B$  be a  $p \times q$  matrix such that  $AB$  and  $BA$  both exist, we must have  $p = n$ , and  $q = m$ . If,  $AB$  and  $BA$  are to be comparable (which is a necessary condition for  $AB$  and  $BA$  to be equal) we must also have  $m = n$ , i.e.,  $A$  must be a square matrix.

A matrix having an inverse is called an *invertible matrix*.

It is quite natural to ask the following questions regarding inverses of matrices :

**Question 1.** Does every square matrix have an inverse?

**Question 2.** In case the answer to question 1 is 'No', how can we test as to whether a given matrix is invertible?

**Question 3.** Can a matrix have more than one inverse?

**Question 4.** In case the answer to question 2 is 'No' how can we proceed to determine the inverse of invertible matrix?

We shall try to answer to above questions one by one.

**Answer to question 1.** Let us consider the matrix.

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

If be any 2-rowed square matrix, we find that

$$AB = BA = 0.$$

We thus find that there cannot be any matrix  $B$  for which  $AB$  and  $BA$  are both equal to  $I^*$ . Therefore  $A$  is not invertible. We, therefore, find that a given square matrix need not have an inverse.

Before taking up question 2, we shall try to answer question 3.

**Answer to questions 3.** As the following theorem shows, if a matrix possesses an inverse, it must be unique.

**Theorem 5.** If  $B$  and  $C$  be inverses of a square matrix,  $A$  then  $B = C$ .

**Proof.** Let  $B$  and  $C$  be inverses of a square matrix  $A$ . Since  $B$  is an inverse of  $A$ , therefore

$$AB = BA = I \quad \dots(1)$$

Again, since  $C$  is an inverse of  $A$ , therefore

$$AC = CA = I \quad \dots(2)$$

From (1) we find that

$$C(AB) = CI = I \quad \dots(3)$$

Also, from (2) we find that

$$(CA)B = IB = B \quad \dots(4)$$

Since  $C(AB) = (CA)B$ , it follows from (3) and (4), that

$$C = B$$

In view of the above theorem it is only proper to talk of *the* inverse of a square matrix rather than taking of *an* inverse.

The inverse of an invertible matrix is denoted by  $A^{-1}$ .

**Answer to question 2.** Suppose a square matrix  $A$  possesses an inverse  $B$ . Then

$$AB = BA = I.$$

Since the determinant of the product of two matrices is equal to the product of their determinants, we find that

$$|AB| = |I|$$

$$\text{i.e., } |A| \times |B| = |I|$$

From the above relation we find that  $|A| \neq 0$ . Thus a *necessary condition for a square matrix to have an inverse is that  $|A| \neq 0$* . We shall now show that this condition is sufficient as well. In

order to do so, let us consider the matrix  $C = \frac{1}{|A|} \text{adj. } A$ .

By using the identity

$$A(\text{adj. } A) = (\text{adj. } A)A = |A|I,$$

We find that 
$$AC = \frac{1}{|A|} (A \text{ adj. } A) = \frac{1}{|A|} \cdot |A|I = I,$$

and 
$$CA = \left( \frac{1}{|A|} \text{adj. } A \right) A = \frac{1}{|A|} (\text{adj. } A) A = \frac{1}{|A|} \cdot |A|I = I.$$

Since  $AC = CA = I$

it follows that  $C$  is the inverse of  $A$ .

From the above discussion we find that a *square matrix possess an inverse if and only if  $|A| \neq 0$* .

This answers question 2. In order to test whether a square matrix possesses an inverse, we have simply to calculate  $|A|$ . If  $|A| = 0$  then  $A$  does not possess an inverse but if  $|A| \neq 0$  then  $A$  possesses an inverse.

**Answer to question 4.** While trying to answer question 2, we saw that if  $|A| \neq 0$ , then the matrix  $\frac{1}{|A|} \text{adj. } A$  is the inverse of  $A$ . This provides one possible answer to question 4.

*If a square matrix  $A$  possesses an inverse  $A^{-1}$ , then in order to find the inverse we compute  $\text{adj } A$ , and multiply it by the scalar  $\frac{1}{|A|}$ . The resulting matrix is the desired inverse of  $A$ .*

**Example 3.** Find the inverse of the matrix

$$\begin{pmatrix} 2 & 5 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 3 \end{pmatrix} \text{ by first computing its adjoint.}$$

**Solution.** The given matrix be denoted by  $A$ .

The co-factors of the elements of the first row of  $A$  are 3,  $-1$ , and 1 respectively.

The co-factors of the elements of the second row of  $A$  are  $-15$ , 6, and  $-5$  respectively.

The co-factors of the elements of the third row are 5,  $-2$ , and 2 respectively.

Therefore 
$$\text{adj. } A = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$$

Also 
$$|A| = 2 \cdot 3 + 5(-1) + 0 \cdot 1 = 1$$

Therefore 
$$A^{-1} = \frac{1}{|A|} \text{adj. } A = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}.$$

#### 4. Use of Elementary Operations to Compute the Inverse of a Matrix

We have already described one method for finding the inverse of an invertible matrix. We shall now describe another method for the same, namely by applying elementary row operations.

Suppose we wish to find the inverse of an  $n \times n$  matrix  $A$ . We consider the identity

$$IA = A.$$

We reduce the matrix  $A$  on the right hand side to triangular form by E-row operations, and apply the same E-row operations to the pre-factor  $I$  on the left hand side. In this manner we get the identity

$$PA = Q,$$

where  $P$  and  $Q$  are some triangular matrices. As our next step, we apply E-row operations on  $Q$  and reduce it to the unit matrix  $I$ . The same E-row operations are, of course, applied to  $P$ . We get the identity  $BA = I$ .

where  $B$  is obtained from  $P$  by E-row operations. The matrix  $B$  is the desired inverse of  $A$ . We shall illustrate the procedure by considering a few examples.

**Example 4.** Find the inver of the matrix.

$$A = \begin{pmatrix} 2 & 3 & -5 \\ -3 & -5 & 9 \\ 1 & 1 & -2 \end{pmatrix}.$$

**Solution.** Consider the identity

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -5 \\ -3 & -5 & 9 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -5 \\ -3 & -5 & 9 \\ 1 & 1 & -2 \end{pmatrix}$$

By performing the E-row operation  $R_2 \rightarrow R_2 + \frac{3}{2}R_1$  on the matrix on right as well as on the pre-factor on the left, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -5 \\ -3 & -5 & 9 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -5 \\ -3 & -\frac{1}{2} & \frac{3}{2} \\ 1 & 1 & -2 \end{pmatrix}$$

Performing  $R_3 \rightarrow R_3 - \frac{1}{2}R_1$  on the matrix on the right as well as on the pre-factor on the left, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & -5 \\ -3 & -5 & 9 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -5 \\ 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Performing  $R_3 \rightarrow R_3 - R_2$  on the matrix on the right as well as on the pre-factor on the left, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -5 \\ -3 & -5 & 9 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -5 \\ 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -1 \end{pmatrix}$$

Performing  $R_1 \rightarrow R_1 - 5R_3$ ,  $R_2 \rightarrow +\frac{3}{2}R_3$  on the matrix on the right as well as on the pre-factor on the left, we have

$$\begin{pmatrix} 11 & 5 & -5 \\ -\frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -5 \\ -3 & -5 & 9 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Performing  $R_1 \rightarrow R_1 + 6R_2$  on the matrix on the right as well as on the pre-factor on the left, we have

$$\begin{pmatrix} 2 & 2 & 4 \\ -\frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -5 \\ -3 & -5 & 9 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Performing  $\frac{1}{2}R_1$ ,  $R_2 \rightarrow -2R_2$ , and  $R_3 \rightarrow -R_3$  on the matrix on the right as well as on the pre-factor on the left, we have

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & -3 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -5 \\ -3 & -5 & 9 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

From the above identity we find that the desired inverse is

$$B = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & -3 \\ 2 & 1 & -1 \end{pmatrix}$$

*Verification.* By actual multiplication we find that

$$AB = BA = I,$$

The above working can often be arranged in a more convenient and compact form as in the following example.

**Example 5.** Find the inverses of the matrix

$$\begin{pmatrix} 4 & -1 & -4 \\ 3 & 0 & -4 \\ 3 & -1 & -3 \end{pmatrix}.$$

**Solution.** We shall first reduce the given matrix to  $I_3$  by E-row operations

$$\begin{pmatrix} 4 & -1 & -4 \\ 3 & 0 & -4 \\ 3 & -1 & -3 \end{pmatrix} \sim \begin{pmatrix} 4 & -1 & -4 \\ 0 & \frac{3}{4} & -1 \\ 0 & -\frac{1}{4} & 0 \end{pmatrix} \left[ R_2 \rightarrow R_2 - \frac{3}{4}R_1, R_3 \rightarrow R_3 - \frac{3}{4}R_1 \right]$$

$$\sim \begin{pmatrix} 4 & -1 & -4 \\ 0 & \frac{3}{4} & -1 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \left[ R_3 \rightarrow R_3 + \frac{1}{3}R_2 \right]$$

$$\sim \begin{pmatrix} 4 & -1 & 0 \\ 0 & \frac{3}{4} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \left[ R_1 \rightarrow R_1 - 12R_3, R_2 \rightarrow R_2 - 3R_3 \right]$$

$$\sim \begin{pmatrix} 4 & 0 & 0 \\ 0 & \frac{3}{4} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \left[ R_1 \rightarrow R_1 + \frac{4}{3}R_2 \right]$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left[ R_1 \rightarrow \frac{1}{4}R_1, R_2 \rightarrow \frac{4}{3}R_2, R_3 \rightarrow -3R_3 \right]$$

The given matrix has been reduced to  $I_3$  by the E-row operation  $R_2 \rightarrow R_2 - \frac{3}{4}R_1, R_3 \rightarrow R_3 - \frac{3}{4}R_1,$

$R_3 \rightarrow R_3 + \frac{1}{3}R_2, R_1 \rightarrow R_1 - 12R_3, R_2 \rightarrow R_2 - 3R_3, R_1 \rightarrow R_1 + \frac{4}{3}R_2, R_1 \rightarrow \frac{1}{4}R_1, R_2 \rightarrow \frac{4}{3}R_2,$  and  $R_3 \rightarrow -3R_3.$

We shall now perform these operations on  $I_3$  in this order. We then have

$$\begin{aligned}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &\sim \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ -\frac{3}{4} & 0 & 1 \end{pmatrix} \left[ R_2 \rightarrow R_2 - \frac{3}{4}R_1, R_3 \rightarrow R_3 - \frac{3}{4}R_1 \right] \\
&\sim \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ -1 & \frac{1}{3} & 1 \end{pmatrix} \left[ R_3 \rightarrow R_3 + \frac{1}{3}R_2 \right] \\
&\sim \begin{pmatrix} 13 & -4 & -12 \\ \frac{9}{7} & 0 & -3 \\ -1 & \frac{1}{3} & 1 \end{pmatrix} \left[ R_1 \rightarrow R_1 - 12R_3, R_2 \rightarrow R_2 - 3R_3 \right] \\
&\sim \begin{pmatrix} 16 & -4 & -10 \\ \frac{9}{4} & 0 & -3 \\ -1 & \frac{1}{3} & 1 \end{pmatrix} \left[ R_1 \rightarrow R_1 + \frac{4}{3}R_2 \right] \\
&\sim \begin{pmatrix} 4 & -1 & -4 \\ 3 & 0 & -4 \\ 3 & -1 & -3 \end{pmatrix} \left[ R_1 \rightarrow \frac{1}{4}R_1, R_2 \rightarrow \frac{4}{3}R_2, R_3 \rightarrow -3R_3 \right]
\end{aligned}$$

The last matrix is the desired invers.

The following theorem is sometimes useful for computing inverses:

**Theorem 6.** *If  $A$  and  $B$  are  $n$ -rowed invertible matrices, then  $A^t$  and  $AB$  are both invertible and*

(a)  $(A^t)^{-1} = (A^{-1})^t$ .

(b)  $(AB)^{-1} = B^{-1}A^{-1}$ .

**Proof.** (a). Since  $AA^{-1} = A^{-1}A = I_n$ , therefore by the reversal law for transposes we have

$$(AA^{-1})^t = (A^{-1}A)^t = I_n^t,$$

i.e.,  $(A^{-1})^t A^t = A(A^{-1})^t = I_n$ .

By the definition of the inverse of a matrix it follows that the matrix  $A^t$  is invertible and its inverse is the matrix  $(A^{-1})^t$ .

(b) Since  $A$  and  $B$  are both  $n$ -rowed invertible matrices, therefore  $A^{-1}$  and  $B^{-1}$  exist and are both  $n$ -rowed square matrices, and consequently  $B^{-1}A^{-1}$  is also an  $n$ -rowed square matrix. Let us denote the matrix  $B^{-1}A^{-1}C$ . We wish to show that  $B^{-1}A^{-1}$  is the inverse of the  $n$ -rowed square matrix  $AB$ . In order to achieve our aim we have to verify that

$$(AB)C = C(AB) = I_n,$$

where  $C$  denotes the matrix  $B^{-1}A^{-1}$ .

$$\begin{aligned} \text{Now } (AB)C &= (AB)(B^{-1}A^{-1}) \\ &= A(BB^{-1})A^{-1} \\ &= AI_n A^{-1} \\ &= AA^{-1} \\ &= I_n \end{aligned}$$

$$\begin{aligned} \text{Also, } C(AB) &= (B^{-1}A^{-1})AB \\ &= B^{-1}(A^{-1}A)B \\ &= B^{-1}I_n B \\ &= B^{-1}B \\ &= I_n \end{aligned}$$

From the above relations we find that  $(AB)C = (AB) = I_n$ .

Hence it follows that the matrix  $AB$  is invertible and its inverse is the matrix, i.e.,

$$(AB)^{-1} = C = B^{-1}A^{-1}.$$

**Remark.** Result (b) of the above theorem is often referred to as the *reversal law of inverses*.

The following example illustrates how the reversal law for inverse can be used for computing the inverses of the product of two invertible matrices.

**Example 6.** Obtain the inverses of the matrices

$$A = \begin{pmatrix} 1 & p & 0 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ 0 & q & 1 \end{pmatrix}$$

and hence that of the matrix

$$C = \begin{pmatrix} 1+pq & p & 0 \\ q & 1+pq & p \\ 0 & q & 1 \end{pmatrix}$$

**Solution.** In the very beginning let us observe the following two facts which will be very useful:

- (i)  $B$  is simply the matrix  $A^t$  with  $p$  replaced by  $q$  every where in  $A^t$ .
- (ii)  $C = AB$ .

**Step 1.** Compute  $A^{-1}$

To compute  $A^{-1}$  we reduce  $A$  to  $I^3$ , by E-row operations.

$$\begin{aligned} \begin{pmatrix} 1 & p & 0 \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix} &\sim \begin{pmatrix} 1 & p & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} [R_2 \rightarrow R_2 - pR_3] \\ &\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} [R_1 \rightarrow R_1 - pR_2] \end{aligned}$$

We find that  $A$  can be reduced to  $I_3$  by applying the E-row operations  $R_2 \rightarrow R_2 - pR_3$  and  $R_1 \rightarrow R_1 - pR_2$ . We now apply these E-row operations to  $I_3$  in this very order. We then get the matrix.

$$\begin{aligned} I_3 &\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -p \\ 0 & 0 & 1 \end{pmatrix} [R_2 \rightarrow R_2 - pR_3] \\ &= \begin{pmatrix} 1 & -p & p^2 \\ 0 & 1 & -p \\ 0 & 0 & 1 \end{pmatrix} [R_1 \rightarrow R_1 - pR_2] \end{aligned}$$

Thus  $A^{-1} = \begin{pmatrix} 1 & -p & p^2 \\ 0 & 1 & -p \\ 0 & 0 & 1 \end{pmatrix} \quad \dots(1)$

From (i) we find that

$$(A^t)^{-1} = (A^{-1})^t = \begin{pmatrix} 1 & 0 & 0 \\ -p & 1 & 0 \\ p^2 & -p & 1 \end{pmatrix}$$

Replacing  $p$  and  $q$  in the above identity throughout and using (i), we have

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -q & 1 & 0 \\ q^2 & -q & 1 \end{pmatrix} \quad \dots(2)$$

By applying the reversal law to (ii) we have

$$\begin{aligned} C^{-1} &= (AB)^{-1} = B^{-1}A^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ -q & 1 & 0 \\ q^2 & -q & 1 \end{pmatrix} \begin{pmatrix} 1 & -p & p^2 \\ 0 & 1 & -p \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -p & p^2 \\ -q & 1 + pq & -p^2q - p \\ q^2 & -pq^2 - q & p^2q^2 + pq + 1 \end{pmatrix} \end{aligned}$$

## Exercise 2

1. Find the inverses of the matrix.

$$\begin{pmatrix} 3 & -15 & 5 \\ -1 & 6 & -2 \\ 1 & -5 & 2 \end{pmatrix}$$

by calculating its adjoint.



2. Show that the matrix.

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{pmatrix}$$

possesses an inverse.

3. Given that  $A = \begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ , calculate  $A^2$  and  $A^{-1}$ .

4. If  $A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -9 & 5 \end{pmatrix}$ ,  $B = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ , verify that  $(AB)^{-1} = B^{-1}A^{-1}$

5. By applying elementary row operations find the inverse of the matrix

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & -3 \\ 2 & 1 & -1 \end{pmatrix}$$

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## RANK OF A MATRIX

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### 1. Introduction

In this lesson we shall introduce the important concept of the rank of a matrix. We shall show that the rank of matrix remains unaltered by elementary operations. This will give us a neat method for finding the rank of matrix.

In the next lesson we shall use the concept of rank to determine whether a given system of linear equation possesses a solution, and if so, then how many linearly independent solutions are there.

### 2. Rank of Matrices

Consider the matrix

$$A = \begin{pmatrix} 2 & -3 & -1 & 4 \\ 7 & 0 & 6 & -5 \\ 1 & 4 & -3 & 2 \end{pmatrix}$$

$A$  is a  $3 \times 4$  matrix. By deleting any one column of  $A$  we can obtain a  $3 \times 3$  matrix. Let us recall that such a matrix is called a sub-matrix of  $A$ . The determinant of such a matrix is called a minor of  $A$ . Thus for example,

$$\begin{vmatrix} 2 & -3 & -1 \\ 7 & 0 & 6 \\ 1 & 4 & -3 \end{vmatrix}, \begin{vmatrix} 2 & -1 & 4 \\ 7 & 6 & -5 \\ 1 & -3 & 2 \end{vmatrix}, \begin{vmatrix} -3 & -1 & 4 \\ 0 & 6 & -5 \\ 4 & -3 & 2 \end{vmatrix}$$

are all minor of  $A$ . Since each of these determinants has 3 rows, these minors are called 3-rowed minors.

By deleting any one row and any two columns of  $A$  we can obtain sub-matrices of the type  $2 \times 2$ . The determinant of any such sub-matrices is called a 2-rowed minor of  $A$ . Thus

$$\begin{vmatrix} 2 & -3 \\ 7 & 0 \end{vmatrix}, \begin{vmatrix} -1 & 4 \\ 6 & -5 \end{vmatrix}, \begin{vmatrix} 0 & 6 \\ 4 & -3 \end{vmatrix}$$

are all 2-rowed minors of  $A$ .

In general, we have the following definitions:

**Definition 1.** Let  $A$  be an  $m \times n$  matrix. The determinant of any  $s \times s$  submatrix of  $A$ , obtained by deleting  $m-s$  rows and  $n-s$  columns of  $A$  is called an  $s$ -rowed minor of  $A$ .

Let us now consider the matrix

$$A = \begin{pmatrix} 2 & 3 & -5 \\ 1 & -7 & 8 \\ 9 & -19 & 9 \end{pmatrix}$$

The matrix  $A$  has only one 3-rowed minor, namely

$$\begin{vmatrix} 2 & 3 & -5 \\ 1 & -7 & 8 \\ 9 & -16 & 9 \end{vmatrix},$$

and this can be seen to be zero.

$A$  has 9 two-rowed minors (why ?) These are not all zero. In fact,

$$\begin{vmatrix} 2 & 3 \\ 1 & -7 \end{vmatrix}$$

is a 2-rowed minor which has the value  $2 \cdot (-7) - 3 \cdot 1 = -17$ .

The fact that every 3-rowed minor of  $A$  is zero and there is at least one 2-rowed minor which is not zero, is usually expressed by saying that the rank of the matrix  $A$  is 2.

Let us now consider the matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It has only one 3-rowed minor, namely

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

and this minor has the value 1. Since  $I$  a 3-rowed non-zero minor, and there are not minors having more than three rows, we say that  $I$  is of rank 3.

In general, we have the following definition :

**Definition 2.** (1) A number  $r$  is said to be the rank of matrix  $A$  if

- (i)  $A$  possesses at least one  $r$ -rowed minor which is different from zero.
  - (ii)  $A$  does not possess any non-zero  $(r + 1)$ -rowed minor
- (2) The rank of every zero matrix is, by definition zero.

The rank of a matrix  $A$  is usually denoted by  $\rho(A)$ .

**Illustrations 1.** If  $A = I_n$ , then  $\rho(A) = n$ .

2. If  $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ , then  $\rho(A) = 0$ .

From the definition of the rank of a matrix we have the following useful criteria for determining the rank of a matrix;

- (i) If all the  $(r + 1)$ -rowed minors of a matrix vanish or if the matrix does not possess any  $(r + 1)$ -rowed minor, then the rank of the matrix must be either less than or at the most equal to  $r$ .
- (ii) If at least one  $r$ -rowed minor of a matrix does not vanish, then the rank of the matrix must be either greater than or at least equal to  $r$ .

It is important (and useful) to note that if every  $(r + 1)$ -rowed minor of a matrix is zero, then every higher order minor is automatically zero (because we can expand every higher order minor ultimately in terms of  $(r + 1)$ -rowed minors).

**Example 1.** Find the rank of the matrix.

$$A = \begin{pmatrix} 2 & -4 & 5 & 6 \\ 3 & 8 & 7 & -1 \end{pmatrix}$$

**Solution.** The given matrix has at least one non-zero minor of order 2, namely

$$\begin{vmatrix} 2 & -4 \\ 3 & 8 \end{vmatrix}$$

(In fact this minor =  $2 \cdot 8 - (-4) \cdot 3 = 28$ ) Actually,  $A$  has 6 2-rowed minors all the of which are non-zero.

Since  $A$  has a non-zero 2-rowed minor, of therefore  $\rho(A) \geq 2$ .

Again, since  $A$  is a  $2 \times 4$  matrix, therefore it does not have any 3-rowed minors. Consequently  $\rho(A) \leq 2$ .

From the relations  $\rho(A) \geq 2$  and  $\rho(A) \leq 2$  we have  $\rho(A) = 2$ .

**Example 2.** Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix},$$

all of  $a$ ,  $b$ , and  $c$  being real numbers.

**Solution.**

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \\ &= (b - c)(c - a)(a - b). \end{aligned}$$

The following different cases arise:

**Case I.**  $a$ ,  $b$ , and  $c$  are all different. If  $a$ ,  $b$ , and  $c$  all the different from each other, then  $|A| \neq 0$ . Therefore  $A$  has a non-zero three-rowed minor, namely

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix},$$

so that  $\rho(A) \geq 3$ .

Also,  $A$  has no higher order minors.

Therefore  $\rho(A) \leq 3$ .

Hence  $\rho(A) = 3$ .

**Case II.** Two of the numbers  $a$ ,  $b$ , and  $c$  are equal but  $a$ ,  $b$ , and  $c$  are not all equal. For the sake of definiteness assume that  $a = b \neq c$ .

In this case  $|A| = 0$  since  $|A|$  is the only 3-rowed minor and this minor vanishes, therefore  $\rho(A) \leq 2$ . Also,  $A$  has atleast one non-zero 2-rowd minor, namely

$$\begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix}$$

Therefore  $\rho(A) \geq 2$ .

Hence  $\rho(A) = 2$ .

In the same way as above we find that  $\rho(A) = 2$  in the cases  $b = c \neq a$  or  $c = a \neq b$ .

Case III.  $a = b = c$ ,

Since  $a = b = c$ , therefore  $|A| = \begin{vmatrix} 1 & 1 & 1 \\ a & a & a \\ a^2 & a^2 & a^2 \end{vmatrix} = 0$  so that the only 3-rowed minor is zero.

Also all the 2-rowed minors of  $A$  vanish. In fact all three columns of  $A$  are identical in this case and therefore all the nine 2-rowed minors of  $A$  vanish. Therefore,  $\rho(A) \leq 1$ .

Also,  $A$  has a non-zero of 1-rowed monor. (In fact these are at least three 1-rowed non-zero minors, namely those which are obtained by deleting the 2nd and 3rd rows and any tow columns of  $A$ ). Therefore

$$\rho(A) \geq 1.$$

Hence  $\rho(A) = 1$ .

### 3. Rank and Elementary Operations

In the two examples discussed above, we has obtained the ranks of the matrices being considered by a straight forward consideration of all the minors. This may not be so easy in all the cases. If we have to find the rank of a  $4 \times 4$  matrix, we may have to consider a fourth order determinants, 16 third order determinants, and perhaps some second order determinants.

By using elementary operations we can conveniently handle the problem of determining the rank of a matrix. The following theorem gives the relation between elementary operations on a matrix and its rank.

**Theorem 1.** *The rank of a matrix remains unaltered by elementary operations.*

**Proof.** We shall prove the theorem for elementary row operations only. The proof for elementary column operations is similar.

Let  $A = [a_{ij}]$  be an  $m \times n$  matrix of rank  $r$ .

**Case I.**  $R_p \leftrightarrow R_q$ .

Let  $B$  be the matrix obtained from  $A$  by the E-row operation  $R_p \leftrightarrow R_q$  and let the rank of  $B$  be  $s$ .

In order to verify that  $\rho(B) = r$ , we shall show that all  $(r + 1)$ -rowed minors of  $B$  vanish and atleast one  $r$ -rowed minor does not vanish.

Let  $B^*$  be any  $(r + 1)$  - rowed sqaue sub-matrix of  $B$ . Since  $B$  has been obtained from  $A$  by the operation  $R_p \leftrightarrow R_q$ , therefore every row of  $B$  is also a row of  $A$  (and every row of  $A$  is a row of  $B$ ). This implies that  $|B^*| = \pm |A^*|$  for some  $(r + 1)$  -rowed submatrix  $A^*$  or  $A$ . But every  $(r + 1)$ -rowed minor of  $A$  is zero (because  $\rho(A) = r$ ). Therefore  $|B^*| = 0$ , and consequently every  $(r + 1)$ -rowed minor of  $B$  is zero. This implies that  $\rho(B) \leq r$ , i.e.,  $s \leq r$ .

Since  $B$  has been obtained from  $A$  by interchanging the  $p^{\text{th}}$  and  $q^{\text{th}}$  rows, therefore we can get  $A$  from  $B$  by applying  $R_p \leftrightarrow R_q$  to  $B$ . By interchanging the roles of  $A$  and  $B$  in the above proof it follows that  $r \leq s$ .

The relations  $s \leq r$  and  $r \leq s$  together yield  $r = s$ .

**Case II.**  $R_p \rightarrow cR_p$  ( $C \neq 0$ )

Let us denote by  $B$  the matrix obtained from  $A$  by the operation  $R_p \rightarrow cR_p$ , and let the rank of  $B$  be  $s$ . Let  $B^*$  be any  $(r + 1)$  - rowed sub-matrix of  $B$  and let  $A^*$  be the correspondingly placed sub-matrix of  $A$ . Let us compare  $|B^*|$  and  $|A^*|$ . Two possibilities arise :

(i)  $p^{\text{th}}$  row of  $B$  is one of the row struck off to obtain  $B^*$  from  $B$ .

In this case  $B^*$  is identical with  $A^*$  and  $|B^*| = |A^*|$ .

(ii)  $p^{\text{th}}$  row of  $B$  is *not* among the rows struck off to obtain  $B^*$  from  $B$ . In this case  $|B^*| = c|A^*|$ , for  $B^*$  and  $A^*$  agree with each other in all rows except one row, and every element of this particular row of  $B^*$  is  $c$  times the corresponding element of  $A^*$ .

Since  $A$  is of rank  $r$ , therefore every  $(r + 1)$  -rowed minor of  $A$  is zero. In particular  $|A^*| = 0$  and in each of the two cases (i) and (ii) above, we have  $|B^*| = 0$ .

Since every  $(r + 1)$  - rowed minor of  $B$  is zero, therefore  $\rho(B) \leq r$ , i.e.,  $s \leq r$ . Also, since  $A$  can be obtained from  $B$  by the E- operation  $R_p \rightarrow C^{-1}R_p$ , therefore by interchanging the roles of  $A$  and  $B$  it follows that  $r \leq s$ . The relations  $s \leq r$  and  $r \leq s$  together yield  $r = s$ .

**Case III.**  $R_p \rightarrow R_p + kR_q$  ( $k \neq 0$ ).

Let us denote by  $B$  the matrix obtained from  $A$  by the E-operation  $R_p \rightarrow kR_q$ , and the rank of  $B$  be  $s$ .

Let  $B^*$  be any  $(r + 1)$  - rowed sub-matrix of  $B$  and let  $A^*$  be the corresponding sub-matrix of  $A$ . Three different possibilities arise :

(i) Neither the  $p^{\text{th}}$  row nor  $q^{\text{th}}$  row of  $B$  has been struck off while obtaining  $B^*$ . In this case  $A^*$  and  $B^*$  differ in only one row (the rows which correspond to  $p^{\text{th}}$  row of  $A$ ) and have all the other rows identical. We have  $|B^*| = |A^*|$ . Since every  $(r + 1)$  - rowed minor of  $A$  is zero, therefore in particular  $|A^*| = 0$ , and consequently  $|B^*| = 0$ .

(ii) The  $p^{\text{th}}$  row of  $B$  has not been struck off while obtaining  $B^*$  from  $B$  but the  $q^{\text{th}}$  row has been struck off.

Let us denote by  $C^*$  the  $(r + 1)$  - rowed matrix obtained from  $A^*$  by replacing  $a_{pj}$  by  $a_{qj}$ . By the property of determinants we have

$$|B^*| = |A^*| + k|C^*|.$$

Observe that  $C^*$  can be obtained from  $A$  by first performing the E-operation  $R_p \rightarrow R_q$  and then striking off those very rows and columns of the new matrix thus obtained as are struck off for obtaining  $B^*$  from  $B$ . This implies that  $|C^*| = \pm 1$  times some  $(r + 1)$  - rowed minor of  $A$ . Since  $\rho(A) = r$ , therefore every  $(r + 1)$  - rowed minor of  $A$  is zero, and so  $|A^*| = 0$ , and  $|C^*| = 0$ . From (1) it follows that  $|B^*| = 0$ .

(iii) The  $p^{\text{th}}$  row of  $B$  is one of those rows which have been struck off while obtaining  $B^*$ . Since  $A$  and  $B$  differ only in the  $p^{\text{th}}$  row, therefore in this case  $B^* = A^*$  and so  $|B^*| = |A^*|$ . Since  $A$  is of rank  $r$ , therefore  $|A^*| = 0$  and consequently  $|B^*| = 0$ . Since  $|B^*| = 0$ , in all the cases (i) - (iii), therefore every  $(r + 1)$  - rowed minor of  $B$  is zero and so  $\rho(B) \leq r$ , i.e.,  $s \leq r$ . Also since  $A$  can be obtained from  $B$  by the E-operation  $R_p \rightarrow R_p - kR_q$ , therefore by interchanging the roles of  $A$  and  $B$  it follows that  $r \leq s$ . Hence  $r = s$ .

We have thus shown that the rank of a matrix remains unaltered by E-row operations, and so remarked in the beginning this completes the proof.

**Example 3.** Reduce the matrix

$$\begin{pmatrix} 3 & -1 & 1 & 3 \\ -1 & -4 & -2 & -7 \\ 2 & 1 & 3 & 0 \\ -1 & -2 & 3 & 0 \end{pmatrix}$$

to triangular form by E-row operations and hence determine its rank.

**Solution.** Let us denote the given matrix by  $A$ . We shall first of all interchange  $R_1$  and  $R_4$  so that we get  $-1$  as the  $(1, 1)$  the entry. This will ultimately simplify the entire working.

$$\begin{aligned} A &= \begin{pmatrix} 3 & -1 & 1 & 3 \\ -1 & -4 & -2 & -7 \\ 2 & 1 & 3 & 0 \\ -1 & -2 & 3 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} -1 & -2 & 3 & 0 \\ -1 & -4 & -2 & -7 \\ 2 & 1 & 3 & 0 \\ 3 & -1 & 1 & 3 \end{pmatrix}, \text{ by } R_1 \leftrightarrow R_4 \\ &\sim \begin{pmatrix} -1 & -2 & 3 & 0 \\ 0 & -2 & -5 & -7 \\ 0 & -3 & 9 & 0 \\ 0 & -7 & 10 & 2 \end{pmatrix}, \text{ by } R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 + 2R_1, R_4 \rightarrow R_4 + 3R_1 \\ &\sim \begin{pmatrix} -1 & -2 & 3 & 0 \\ 0 & -2 & -5 & -7 \\ 0 & 0 & \frac{33}{2} & \frac{21}{2} \\ 0 & 0 & \frac{55}{2} & \frac{55}{2} \end{pmatrix}, \text{ by } R_3 \rightarrow R_3 - \frac{3}{2}R_2, R_4 \rightarrow R_4 - \frac{7}{2}R_2 \\ &\sim \begin{pmatrix} -1 & -2 & 3 & 0 \\ 0 & -2 & -5 & -7 \\ 0 & 0 & \frac{33}{2} & \frac{21}{2} \\ 0 & 0 & 0 & 10 \end{pmatrix}, \text{ by } R_4 \rightarrow R_4 - \frac{5}{3}R_3 \end{aligned}$$

If we denote the above matrix by  $B$ , then we see that  $B = 330 \neq 0$ , so that rank of  $B = 4$ .

But  $A$  and  $B$  have the same rank.

Therefore  $A$  is of rank 4.

**Example 4.** Find the rank of the matrix 
$$\begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 \\ 14 & 15 & 16 & 17 & 18 \end{pmatrix}.$$

**Solution.** Let us denote the given matrix by  $A$ . Observe that if we subtract the element of the first row from the corresponding elements of the second row, then the second row will consist of ones only.

In fact by  $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1, R_4 \rightarrow R_4 - R_1, R_5 \rightarrow R_5 - R_1$ , and then  $R_3 \rightarrow \frac{1}{2}R_3, R_4 \rightarrow \frac{1}{7}R_4,$

$R_5 \rightarrow \frac{1}{12}R_5$ , we find that

$$A \sim \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ by } C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1, C_4 \rightarrow C_4 - C_1, C_5 \rightarrow C_5 - C_1]$$

$$\sim \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} [\text{by } C_3 \rightarrow C_3 - 2C_2, C_4 \rightarrow C_4 - 3C_2, C_5 \rightarrow C_5 - 4C_2]$$

Denoting the last matrix by  $B$ , we find that every 3-rowed minor of  $B$  is zero, and  $B$  has at least one non-zero two-rowed minor, namely  $\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}$ .

There  $\rho(B) = 2$ .

But  $\rho(A) = \rho(B)$ , so that  $\rho(A) = 2$ .

**Remark.** Observe that in the above example we did not proceed in the normal way for triangular reduction.

**Example 5.** If  $x, y, z$  are all unequal, show by using elementary operations or otherwise that the matrix

$$A = \begin{pmatrix} 0 & x-y & x-z & y+z \\ y-x & 0 & y-z & x+z \\ z-x & z-y & 0 & x+y \\ y+z & z+z & x+y & 0 \end{pmatrix}$$

is of rank 2.



**Solution.**

$$A = \begin{pmatrix} 0 & x-y & x-z & y+z \\ y-x & 0 & y-z & x+z \\ z-x & z-y & 0 & x+y \\ y+z & z+x & x+y & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} x-y & x-y & x-y & y-x \\ y-x & 0 & y-z & x+z \\ z-x & z-y & 0 & x+y \\ y+z & z+x & x+y & 0 \end{pmatrix} [\text{by } R_1 \rightarrow R_1 - R_2]$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & -1 \\ y-x & 0 & y-z & x+z \\ z-x & z-y & 0 & x+y \\ y+z & x-y & x+y & 0 \end{pmatrix}, \left[ \text{by } R_1 \rightarrow \frac{1}{x-y} R_1 \right]$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ y-x & x-y & x-z & y+z \\ z-x & x-y & x-z & y+z \\ y+z & x-y & x-z & y+z \end{pmatrix}, [\text{by } C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1, C_4 \rightarrow C_4 + C_1]$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x-y & x-z & y+z \\ 0 & x-y & x-z & y+z \\ 0 & x-y & x-z & y+z \end{pmatrix}, [\text{by } R_2 \rightarrow R_2 - (y-z)R_1, R_3 \rightarrow R_3 - (z-x)R_1, R_4 \rightarrow R_4 - (y+z)R_1]$$

Two different cases arise according as  $y+z \neq 0$  or  $y+z = 0$ .

**Case I.**  $y+z \neq 0$ .

$$A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \left[ \text{by } C_2 \rightarrow \frac{1}{x-y} C_2, C_3 \rightarrow \frac{1}{x-z} C_3, C_4 \rightarrow \frac{1}{y+z} C_4 \right]$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, [\text{by } C_3 \rightarrow C_3 - C_2, C_4 \rightarrow C_4 - C_2]$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, [\text{by } R_3 \rightarrow R_3 - R_2, R_4 \rightarrow R_4 \sim R_2]$$

Since the matrix  $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$  is of rank 2, therefore it follows that  $\rho(A) = 2$

**Case II.**  $y + z = 0$

$$\begin{aligned} A &\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x-y & x-z & 0 \\ 0 & x-y & x-z & 0 \\ 0 & x-y & x-z & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \left[ \text{by } C_2 \rightarrow \frac{1}{x-y}C_2, C_3 \rightarrow \frac{1}{x-z}C_3 \right] \\ &\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \left[ \text{by } R_3 \rightarrow R_3 - R_2, R_4 \rightarrow R_4 - R_2 \right] \\ &\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \left[ \text{by } C_3 \rightarrow C_3 - C_2 \right] \end{aligned}$$

which is of rank 2.

Hence  $\rho(A) = 2$  in this case also.

Thus  $\rho(A) = 2$

**Example 6.** Find the rank of the matrix

$$\begin{pmatrix} 0 & r & -q & x \\ -r & 0 & p & y \\ q & -p & 0 & z \\ -x & -y & -z & 0 \end{pmatrix},$$

where  $px + qy + rz = 0$ , and all of  $p$ ,  $q$ , and  $r$  are positive real numbers.

**Solution.** Let the given matrix be denoted by  $A$ .

$$A = \begin{pmatrix} 0 & r & -q & x \\ -r & 0 & p & y \\ q & -p & 0 & z \\ -x & -y & -z & 0 \end{pmatrix},$$

$$\begin{aligned}
& \sim \begin{pmatrix} 0 & r & -q & px \\ -r & 0 & p & py \\ q & -p & 0 & py \\ q & -p & -2 & 0 \end{pmatrix}, [\text{by } C_4 \rightarrow pC_4, \text{ since } p \neq 0] \\
& \sim \begin{pmatrix} 0 & r & -q & px+qy \\ -r & 0 & p & 0 \\ q & -p & 0 & pz \\ -x & -y & -z & yz \end{pmatrix}, [\text{by } C_4 \rightarrow C_4 - yC_3] \\
& \sim \begin{pmatrix} 0 & r & -q & px+qy+rz \\ -r & 0 & p & 0 \\ q & -p & 0 & 0 \\ -x & -y & -z & 0 \end{pmatrix}, [\text{by } C_4 \rightarrow C_4 + zC_2] \\
& \sim \begin{pmatrix} 0 & r & -q & 0 \\ -r & 0 & p & 0 \\ q & -p & 0 & 0 \\ -x & -y & -z & 0 \end{pmatrix}, \text{since } px+qy+rz=0. \\
& \sim \begin{pmatrix} 0 & r & -qr & 0 \\ -r & 0 & pr & 0 \\ q & -p & 0 & 0 \\ -x & -y & -rz & 0 \end{pmatrix}, [\text{by } C_3 \rightarrow rC_3] \\
& \sim \begin{pmatrix} 0 & r & 0 & 0 \\ -r & 0 & pr & 0 \\ q & -p & -pq & 0 \\ -x & -y & -qy-rz & 0 \end{pmatrix}, [C_3 \rightarrow C_2 + qC_2] \\
& \sim \begin{pmatrix} 0 & r & 0 & 0 \\ -r & 0 & 0 & 0 \\ q & -p & 0 & 0 \\ -x & -y & -(px+qy+rz) & 0 \end{pmatrix}, [by } C_3 \rightarrow C_3 + qC_2] \\
& \sim \begin{pmatrix} 0 & r & 0 & 0 \\ -r & 0 & 0 & 0 \\ q & -p & 0 & 0 \\ -x & -y & 0 & 0 \end{pmatrix}, \text{since } px+qy+rz=0. \\
& = B(\text{say})
\end{aligned}$$

Now every 3-rowed minor of the matrix  $B$  vanishes, and there is atleast one non-zero 2-rowed minor namely  $\begin{vmatrix} 0 & r \\ -r & 0 \end{vmatrix}$ , therefore the rank of  $B$  is 2. Since  $A \sim B$ , therefore  $\rho(A) = \rho(B) = 2$ .

**Example 7.** If the number  $a$ ,  $b$ , and  $c$  are all different from each other, and the numbers,  $x$ ,  $y$ ,  $z$  are also all different from each other show that the matrix

$$A = \begin{pmatrix} 1 & a & x & ax \\ 1 & b & y & by \\ 1 & c & z & cz \end{pmatrix}$$

is of rank 3.

**Solution.**

$$\begin{aligned} A &= \begin{pmatrix} 1 & a & x & ax \\ 1 & b & y & by \\ 1 & c & z & cz \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & a & x & ax \\ 0 & b-a & y-x & by-ax \\ 0 & c-a & z-x & cz-ax \end{pmatrix}, [\text{by } R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1] \\ &\sim \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & b-a & y-x & by-ax \\ 0 & c-a & z-x & cz-ax \end{pmatrix}, [\text{by } C_2 \rightarrow C_2 - aC_1, C_3 \rightarrow C_3 - xC_1, C_4 \rightarrow C_4 - axC_1] \\ &\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b-a & y-x & b(y-x) \\ 0 & c-a & z-x & c(z-x) \end{pmatrix}, [\text{by } C_4 \rightarrow C_4 - xC_2] \\ &\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b-a & y-x & 0 \\ 0 & c-a & z-x & (c-b)(z-x) \end{pmatrix}, [\text{by } C_4 \rightarrow C_4 - bC_3] \\ &= B(\text{say}). \end{aligned}$$

Let us first of all consider the rank of  $B$ . since  $B$  is a  $3 \times 4$  matrix,  $\rho(B) \leq 3$ . Also, since  $B$  has a 3-rowed non-zero minor, namely  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & y-x & 0 \\ 0 & z-x & (c-b)(z-x) \end{pmatrix}$ , therefore  $r(B) \leq 3$ .

From the inequalities  $\rho(B) \leq 3$  and  $\rho(B) \geq 3$  it follows that  $\rho(B) = 3$ .

Also, since the rank of a matrix remains unaltered by elementary operations, therefore  $\rho(A) = \rho(B) = 3$ .

Hence the rank of the given matrix is 3.

### Exercise

1. Find the rank of the matrix

$$\begin{pmatrix} 3 & -1 & 2 & -6 \\ 6 & -2 & 4 & -9 \end{pmatrix}.$$

2. Find the rank of the matrix

$$\begin{pmatrix} -1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

3. Find the rank of the matrix.

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{pmatrix}.$$

4. Reduce the matrix

$$\begin{pmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

to triangular form and hence find its rank.

5. Find the rank of the matrix.

$$\begin{pmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{pmatrix}.$$



matrix,  $X$  is an  $n \times 1$  matrix, and  $B$  is an  $m \times 1$  matrix, then we can write (2) as

$$AX = B$$

The matrix  $A$  is called the co-efficient matrix, and  $[A, B]$  is called the *augmental matrix*, where  $[A, B]$  stands for the  $m \times (n + 1)$  matrix obtained by adjoining one column, namely  $(b_1, b_2) \dots b_m)^t$  to the  $m \times n$  matrix  $A$ .

We shall confine our study to the following aspects :

I. Application of the inverse of a matrix to study a system of  $n$  equations  $AX = B$  in  $n$ -unknowns. We shall show that if the matrix  $A$  possesses an inverse, then the system has the unique solution which is given by  $X = A^{-1}B$ . As a corollary of this result we shall show that if the matrix  $A$  is invertible, then the system of equations  $AX = 0$  does not possess any non-zero solution.

II. We shall use elementary row operations to put  $AX = B$  in a simplified form from which the solution of the system can be written easily.

III. We shall apply the concept of rank of a matrix to discuss the existence and uniqueness of solutions of  $AX = B$ . We shall show that if  $A$  be  $m \times n$  matrix of rank  $r$ , then (i) the general solution of the system of homogeneous equations  $AX = 0$  contains  $n-r$  arbitrary constants, and (ii) the system of non-homogeneous equations  $AX = B$  possess a solution if and only if the matrix  $[A, B]$  is also of rank  $r$ . In case this condition is satisfied, the solution contains  $n-r$  arbitrary constant.

## 2. Use of Inverse of Matrix to Solve a System of Linear Equations

Consider a system of  $n$  linear equations

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n, \end{aligned} \right\} \dots(1)$$

in  $n$  unknowns  $x_1, x_2, \dots x_n$ .

We can write the above system of equation as

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix} \dots(2)$$

If we write

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix},$$

and

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{pmatrix},$$

so that  $A$  is an  $n \times n$  matrix,  $X$  is an  $n \times 1$  matrix, and  $B$  is an  $n \times 1$  matrix then we can write (2) as

$$AX = B \quad \dots(3)$$

If  $A$  is a non-singular matrix, so that  $A^{-1}$  exists, we pre-multiply (3) throughout by  $A^{-1}$ , as that

$$A^{-1}(AX) = A^{-1}B$$

$\Rightarrow$

$$(A^{-1}A)X = A^{-1}B$$

$\Rightarrow$

$$IX = A^{-1}B$$

$\Rightarrow$

$$X = A^{-1}B. \quad \dots(4)$$

From (4) we find that if the system of equations (3) has a solution, it is given by (4). By substituting  $X = A^{-1}B$  in (3) we find that

$AX = A(A^{-1}B) = (AA^{-1})B = IB = B$ , showing that  $X = A^{-1}B$  is indeed a solution. Thus we have the following result.

**Corollary.** *If  $A$  is an  $n$ -rowed non-singular matrix, the system of homogeneous linear equations.  $AX = 0$  has  $X = 0$  as its only solution.*

**Example 1.** *Show that the system of equations*

$$4x + 3y + 3z = 1$$

$$-x - z = 2$$

$$-4x - 4y - 3z = 3$$

*has a unique solution, and obtain it by first computing the inverse of the co-efficient matrix.*

**Solution.** The given system of equations can be written as

$$\begin{pmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

or as  $AX = B$ .

where

$$A = \begin{pmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

$$|A| = \begin{vmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -1 \\ -4 & -3 \end{vmatrix} - 3 \begin{vmatrix} -1 & -1 \\ -4 & -3 \end{vmatrix} + 3 \begin{vmatrix} -1 & 0 \\ -4 & -4 \end{vmatrix}$$



$$\begin{aligned}
&= 4(-4) - 3 \cdot (-1) + 3 \cdot 4 \\
&= -1.
\end{aligned}$$

Since  $|A| \neq 0$ , the matrix is invertible. Since the co-efficient matrix is invertible, therefore the given system of equations possesses a unique solution.

We shall now find  $A^{-1}$ .

Now adj. 
$$A = \begin{pmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj. } A$$

$$= \begin{pmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{pmatrix}$$

$$X = A^{-1}B$$

$$= \begin{pmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

$$= \begin{pmatrix} 19 \\ -4 \\ -21 \end{pmatrix}$$

Therefore  $x = 19$ ,  $y = -4$ ,  $z = -21$  is the desired solution.

**Verification.** When  $x = 19$ ,  $y = -4$ ,  $z = -21$ ,

$$4x + 3y + 3z = 4(19) + 3(-4) + 3(-21) = 1,$$

$$-x - z = -19 + 21 = 2$$

$$-4x - 4y - 3z = -4(19) - 4(-4) - 3(-21) = 3.$$

**Example 2.** Show, by considering the co-efficient matrix, that the system of equations

$$x + 2y - z = 0,$$

$$-4x - 7y + 4z = 0,$$

$$-4x - 9y + 5z = 0,$$

does not possess a non-zero solution.

**Solution.** The co-efficient matrix of the given system of equation of given by

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -9 & 5 \end{pmatrix}$$

$$\begin{aligned}
|A| &= \begin{vmatrix} -7 & 4 \\ -9 & 5 \end{vmatrix} - 2 \begin{vmatrix} -4 & 4 \\ -4 & 5 \end{vmatrix} - \begin{vmatrix} -4 & -7 \\ -4 & -9 \end{vmatrix} \\
&= 1 - 2(-4) - 8 \\
&= 1.
\end{aligned}$$

Since  $|A| \neq 0$ , therefore  $A$  is invertible. Since  $A^{-1}$  exists, the given system of equations does not possess a non-zero solution.

### 3. Use of Elementary Row Operation to Solve a System of Linear Equations

Consider the system of  $m$  linear equations

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \dots &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m, \end{aligned} \right\} s \quad \dots(1)$$

By writing

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{pmatrix}$$

we can write (1) as

$$AX = B,$$

where  $A = [a_{ij}]$  is an  $m \times n$  matrix,  $X = (x_1, x_2, \dots, x_n)^t$  is an  $n \times 1$  matrix, and  $B = (b_1, b_2, \dots, b_n)^t$  is an  $m \times 1$  matrix.

The system of equations (1) remains unaltered by the following operations :

- (i) Interchange of any two equations;
- (ii) Multiplication of both sides of an equation by non-zero number  $k$ ;
- (iii) addition of  $k$  times an equation to another equation.

The above equations are equivalent to the following elementary row operations on the matrices  $A$  and  $B$ .

- (i) interchange of two rows.
- (ii) multiplying a row of  $A$  and the corresponding row of  $B$  by a non-zero number  $k$ ;
- (iii) adding  $k$  times an equation to another equation.

Let us recall that :

- (a) A matrix can be reduced to triangular form by E-row operations.
- (b) An E-row operation on the product of two matrices is equivalent to the same E-row operation on the pre-factor.

In view of the all the facts stated above we find that in order to solve a given system of linear equation we may proceed as follows :

- (1) Write the given system in matrix notation as  $AX = B$ .
- (2) Reduce  $A$  to a triangular matrix say  $A^*$  by E-row operations.
- (3) Keep on applying the E-row operations in (2) above to the matrix  $B$  in the same order as in (2) above.
- (4) Suppose the result of (2) and (3) gives us the equation  $A^* X = B^*$  where  $A^*$  is a triangular matrix. By writing  $A^* X = B^*$  as a system of linear equations, we shall arrive at a system of equations which can be solved easily. Of course, it can be happen that the system does not possess a solution. This will happen if one of the equations in  $A^* X = B^*$  is of the form

$$0 \cdot x_1 + 0 \cdot x_2 + \dots + 0 \cdot x_n = b,$$

for some non-zero number  $b$ .

We shall illustrate the method with the help of examples.

**Example 3.** Solve the system of equations

$$2x - 5y + 7z = 6,$$

$$x - 3y + 4z = 3,$$

$$3x - 8y + 11z = 11.$$

**Solution.** Let  $A = \begin{pmatrix} 2 & -5 & 7 \\ 1 & -3 & 4 \\ 3 & -8 & 11 \end{pmatrix}, B = \begin{pmatrix} 6 \\ 3 \\ 11 \end{pmatrix}$

and assume that there exists a matrix  $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ , such that  $AX = B$

Now  $\begin{pmatrix} 2 & -5 & 7 \\ 1 & -3 & 4 \\ 3 & -8 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 11 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & -3 & 4 \\ 2 & -5 & 7 \\ 3 & -8 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 11 \end{pmatrix}$ , by  $R_1 \leftrightarrow R_2$

$\Rightarrow \begin{pmatrix} 1 & -3 & 4 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ , by  $R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 3R_1$

$$\Rightarrow \begin{pmatrix} 1 & -3 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}, \text{ by } R_3 \rightarrow R_3 - R_2$$

$$\Rightarrow \begin{aligned} x - 3y + 4z &= 3, \\ y - z &= 0, \\ 0 &= 2. \end{aligned}$$

Since the conclusion  $0 = 2$  is false, therefore our assumption that for some  $X$ ,  $AX = B$ , is false. Consequently there is no  $X$  for which  $AX = B$ , that is, the given system of equations has no solution.

**Remark.** A system of equations having no solution is said to be *inconsistent*.

**Example 4.** Solve the system of linear equations

$$\begin{aligned} x - 2y + 3z &= 6, \\ 3x + y - 4z &= -7, \\ 5x - 3y + 2z &= 5. \end{aligned}$$

**Solution.** Let  $A = \begin{pmatrix} 1 & -2 & 3 \\ 3 & 1 & -4 \\ 5 & -3 & 2 \end{pmatrix}$ ,  $B = \begin{pmatrix} 6 \\ -7 \\ 5 \end{pmatrix}$ , and assume that there exists a matrix  $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ , such

that  $AX = B$ .

$$\text{Then} \quad \begin{pmatrix} 1 & -2 & 3 \\ 3 & 1 & -4 \\ 5 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -7 \\ 5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -2 & 3 \\ 0 & 7 & -13 \\ 0 & 7 & 13 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -25 \\ -25 \end{pmatrix}, \text{ by } R_2 \rightarrow R_2 - 3R_1, R_3 \rightarrow R_3 - 5R_1.$$

$$\Rightarrow \begin{pmatrix} 1 & -2 & 3 \\ 0 & 7 & -13 \\ 0 & 7 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -25 \\ 0 \end{pmatrix}, \text{ by } R_3 \rightarrow R_3 - R_2$$

$$\Rightarrow \begin{cases} x - 2y + 3z = 6, \\ 7y - 13z = -25 \end{cases}$$

Form the second of above equations, we have  $y = \frac{13}{7}z - \frac{25}{7}$ . Substituting this value of  $y$  in the

first equation, we have  $x = 2y - 3z + 6 = \left(\frac{13}{7}z - \frac{25}{7}\right) - 3z + 6 = \frac{5}{7}z - \frac{8}{7}$ .

Therefore we have

$$x = \frac{5}{7}z - \frac{8}{7}$$

$$y = \frac{13}{7}z - \frac{25}{7},$$

$$z = z,$$

*i.e.*,  $x = \frac{5}{7}k - \frac{8}{7}, y = \frac{13}{7}k - \frac{25}{7}, z = k$ , where  $k$  is any real number. By actual substitution we find that these values satisfy the given equations.

Hence all the solutions of the given system of equations are

$$x = \frac{5}{7}k - \frac{8}{7}, y = \frac{13}{7}k - \frac{25}{7}, z = k,$$

where  $k$  is any real number

**Example 5.** Solve the following system of linear equations :

$$2x - 3y + z = 0,$$

$$x + 2y - 3z = 0,$$

$$4x - y - 2z = 0.$$

**Solution.** Let  $\begin{pmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & -2 \end{pmatrix}$ , and assume that there exists  $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ , such that  $AX = O$ , where

$O$  stand for the zero-matrix of type  $3 \times 1$ .

$$\text{Then } \begin{pmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = O$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ 4 & -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = O, \text{ by } R_1 \leftrightarrow R_2$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & -7 & 7 \\ 0 & -9 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = O, \text{ by } R_2 \leftrightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 4R_1$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & -9 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = O, R_2 \rightarrow -\frac{1}{7}R_2$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = O, R_3 \rightarrow R_3 + 9R_2$$

$$\Rightarrow \begin{cases} x + 2y - 3z = 0 \\ y - z = 0 \\ z = 0 \end{cases}$$

$$\Rightarrow x = 0, y = 0, z = 0.$$

Thus we find that if the given system possesses a solution, it must be given by

$$x = 0, y = 0, z = 0.$$

Also, by actual substitution it follows that  $x = y = z = 0$  is a solution.

Hence  $x = y = z = 0$  is the only solution of the given system.

**Example 6.** Solve the following system of equations :

$$x + 3y + 4z + 7w = 0,$$

$$2x + 4y + 5z + 8w = 0,$$

$$3x + y + 3z + 3w = 0,$$

**Solution.** Let  $A = \begin{pmatrix} 1 & 3 & 4 & 7 \\ 2 & 4 & 5 & 8 \\ 3 & 1 & 2 & 3 \end{pmatrix}$  and assume that there exists a matrix  $X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$  such that  $\mathbf{AX}$

$= O$ , where  $O$  stands for the zero-matrix of type  $4 \times 1$ .

$$\text{Then } \begin{pmatrix} 1 & 3 & 4 & 7 \\ 2 & 4 & 5 & 8 \\ 3 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = O$$

$$\Rightarrow \begin{pmatrix} 1 & 3 & 4 & 7 \\ 0 & -2 & -3 & -6 \\ 0 & -8 & -10 & -18 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = O, \text{ by } R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 3R_1$$

$$\Rightarrow \begin{pmatrix} 1 & 3 & 4 & 7 \\ 0 & -2 & -3 & -6 \\ 0 & 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = 0, R_3 \rightarrow R_3 - 4R_2$$

$$\Rightarrow \begin{cases} x + 3y - 4z + 7w = 0 \\ -2y - 3z - 6w = 0 \\ 2z + 6w = 0 \end{cases}$$

$$\Rightarrow z = -3w, y = \frac{3}{2}w, x = \frac{1}{2}w$$

$$\Rightarrow x = \frac{1}{2}k, y = \frac{3}{2}k, z = -3k, w = k,$$

where  $k$  is any real number. Thus we find that if the given system possesses a solution it must be given by

$$x = \frac{1}{2}k, y = \frac{3}{2}k, z = -3k, w = k, \text{ where } k \text{ is any real number.}$$

By actual substitution we find that (1) is indeed a solution of the given system.

Hence all the solutions of the given system of equations are given by  $x = \frac{1}{2}k$ ,  $y = \frac{3}{2}k$ ,  $z = -3k$ ,  $w = k$ , where  $k$  is any real number.

**Example 7.** Solve the following system of equations :

$$\begin{aligned}x - 3y + 2z &= 0, \\7x - 21y + 14z &= 0, \\-3x + 9y - 6z &= 0.\end{aligned}$$

**Solution.** Let  $A = \begin{pmatrix} 1 & -3 & 2 \\ 7 & -21 & 17 \\ -3 & 9 & -6 \end{pmatrix}$  and assume that there exists a matrix  $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ , such that  $AX = O$ , where  $O$  is the zero-matrix of type  $3 \times 1$ .

$$\text{Then } \begin{pmatrix} 1 & -3 & 2 \\ 7 & -21 & 14 \\ -3 & 9 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = O$$

$$\Rightarrow \begin{pmatrix} 1 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0, \text{ by } R_2 \rightarrow R_2 - 7R_1, R_3 \rightarrow R_3 + R_1$$

$$\Rightarrow x - 3y + 2z = 0$$

$$\Rightarrow x = 3y - 2z, y = y, z = z$$

$$\Rightarrow x = 3k_1 - 2k_2, y = k_1, z = k_2, \quad \dots(1)$$

where  $k_1, k_2$  are any real numbers.

Thus we find that if there is a solution of the given system, it must be (1).

Again, by actually substituting  $x = 3k_1 - 2k_2$ ,  $y = k_1$ ,  $z = k_2$  in the given system of equations, we find that

$$\begin{aligned}x - 3y + 2z &= (3k_1 - 2k_2) - k_1 + 2k_2 = 0, \\7x - 21y + 14z &= 7(3k_1 - 2k_2) - 21k_1 + 12k_2 = 0, \\-3x + 9y - 6z &= -3(3k_1 - 2k_2) + 9k_1 - 6k_2 = 0,\end{aligned}$$

so that (1) is indeed a solution. Therefore we conclude that

$x = 3k_1 - 2k_2$ ,  $y = k_1$ ,  $z = k_2$ , where  $k_1, k_2$  are real numbers, are all the solutions of the given system of equations.

#### 4. Application of Rank of a Matrix to a System of Linear Equations

Let us consider a system of  $m$  linear equations in  $n$  variables:

$$AX = B,$$

where  $A$  is an  $m \times n$  matrix,  $B$  is an  $m \times 1$  matrix, and  $X$  is an  $n \times 1$  matrix. It can be shown that the above system of equations is consistent if and only if the matrices  $A$  and  $[A, B]$  have the same rank. Further more, if the common rank be  $r$ , the general solution contains  $n-r$  arbitrary constants.

We shall not prove the above result, but will content ourselves with the following illustrative examples.

**Example 8.** Show, without actually solving, that the following system of equations is inconsistent.

$$\begin{aligned}x - y + z &= -1, \\2x + y - 4z &= -1, \\6x - 7y + 8z &= 7.\end{aligned}$$

**Solution.** The given system of equations can be written as

$$AX = B$$

where

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -4 \\ 6 & -7 & 8 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} -1 \\ 1 \\ 7 \end{pmatrix}.$$

Let us find the ranks of the matrices  $A$  and  $[A, B]$

$$\begin{aligned}A &= \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -4 \\ 6 & -7 & 8 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -6 \\ 0 & -1 & 2 \end{pmatrix}, \text{ by } R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 6R_1 \\ &\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & -2 \end{pmatrix}, \text{ by } R_2 \rightarrow \frac{1}{3}R_2 \\ &\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \text{ by } R_3 \rightarrow R_3 + R_2\end{aligned}$$

Since

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \text{ is of rank 2,}$$

therefore  $\rho(A) = 2$

Also,

$$\begin{aligned}[A, B] &= \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & 1 & -4 & -1 \\ 6 & -7 & 8 & 7 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 3 & -6 & 1 \\ 0 & -1 & 2 & 13 \end{pmatrix}, \text{ by } R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 6R_1 \\ &\sim \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & \frac{40}{3} \end{pmatrix}, \text{ by } R_3 \rightarrow R_3 + \frac{1}{3}R_2 \\ &\sim \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ by } R_3 \rightarrow \frac{3}{40}R_3\end{aligned}$$



$$\sim \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{by } R_1 \rightarrow R_1 + R_3, R_2 \rightarrow R_2 - R_3$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{by } C_2 \rightarrow C_2 + C_1, C_3 \rightarrow C_3 - C_1$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{by } C_2 \rightarrow \frac{1}{3}C_2, C_3 \rightarrow \frac{-1}{6}C_3$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{by } C_3 \rightarrow C_3 - C_2$$

showing that the rank of augmented matrix is 3.

Since the ranks of the co-efficient matrix and augmented matrix are not equal, the given system of equations is inconsistent.

$$3x + 4y - 6z + w = 7$$

$$x - 2y + 3z + 2w = -1$$

$$x - 3y + 4z - w = -2$$

$$5x - y + z - 2w = 4.$$

**Solution.**

**A**

**B**

$$? = \begin{pmatrix} 3 & 4 & -6 & 1 & : & 7 \\ 1 & -2 & 3 & -2 & : & -1 \\ 1 & -3 & 4 & -1 & : & -2 \\ 5 & -1 & 1 & -2 & : & 4 \end{pmatrix}$$

$$? \sim \begin{pmatrix} 1 & -2 & 3 & -2 & : & -1 \\ 3 & 4 & -6 & 1 & : & 7 \\ 1 & -3 & 4 & -1 & : & -2 \\ 5 & -1 & 1 & -2 & : & 4 \end{pmatrix}, \text{by } R_1 \leftrightarrow R_2$$

$$? \sim \begin{pmatrix} 1 & -2 & 3 & -2 & : & -1 \\ 0 & 10 & -15 & 7 & : & 10 \\ 0 & -1 & 1 & 1 & : & -1 \\ 0 & 9 & -14 & 8 & : & 9 \end{pmatrix}, \text{by } R_2 \rightarrow R_2 - 3R_1, R_3 \rightarrow R_3 - R_1, R_4 \rightarrow R_4 - 5R_1$$

$$? \sim \begin{pmatrix} 1 & -2 & 3 & -2 & : & -1 \\ 0 & 10 & -15 & 7 & : & 10 \\ 0 & 0 & -\frac{1}{2} & \frac{17}{10} & : & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{17}{10} & : & 0 \end{pmatrix}, R_3 \rightarrow R_3 + \frac{1}{10}R_2, R_4 \rightarrow R_4 - \frac{9}{10}R_2$$

$$? \sim \begin{pmatrix} 1 & -2 & 3 & -2 & : & -1 \\ 0 & 10 & -15 & 7 & : & 10 \\ 0 & 0 & -\frac{1}{2} & \frac{17}{10} & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{pmatrix}, \text{ by } R_4 \rightarrow R_4 - R_3.$$

From the above we find that the ranks of  $A$  and  $[A, B]$  are both equal to 3. Since the co-efficient matrix and augment both have the same rank, therefore the given system of equations is consistent.

**Remark.** Observe that in the above example we have computed the ranks of  $A$  and  $[A, B]$  simultaneously. This has meant considerable simplification in workings.

### Exercise

1. Does the following system of equations possess a unique common solution?

$$x + 2y + 3z = 6,$$

$$2x + 4y + z = 7,$$

$$3x + 2y + 9z = 10.$$

If so, find it.

2. Show that  $x = y = z = 0$  is the only common solution of the following system of equations:

$$2x + 3y + 4z = 0,$$

$$x - 2y - 3z = 0,$$

$$5x + y - 8z = 0.$$

3. Solve the system of equations :

$$x + y + z = 3,$$

$$3x - 5y + 2z = 8,$$

$$5x - 3y + 4z = 14.$$

4. Solve the following system of equations :

$$x - 3y + 2z = 0,$$

$$7x - 21y + 14z = 0,$$

$$-3x + 9y - 6z = 0.$$

5. Which of the following system of equations are consistent ?

(a)  $x - 4y + 7z = 8,$

$$3x + 8y - 2z = 6,$$

$$7x - 8y + 26z = 31.$$

(b)  $x - y + 2z = 4,$

$$3x + y + 4z = 6,$$

$$x + y + z = 1.$$

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## THE CHARACTERISTIC EQUATION OF A MATRIX

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### 1. Introduction

In this lesson we shall study the important concept of characteristic equation of a square matrix.

The characteristic equation of a square matrix has many applications. It can be used to compute power of a square matrix, and inverse of a non-singular matrix. It is also useful in simplifying matrix polynomials.

We shall begin with some definitions.

**Definition.** Let  $A = [A_{ij}]$  be an  $n \times n$  matrix. The matrix  $A - \lambda I$ , where  $\lambda$  is a scalar and  $I$  is the  $n$ -rowed unit matrix, is called the characteristic matrix of  $A$ . The determinant  $|A - \lambda I|$  is called the characteristic polynomial of  $A$ . The equation  $|A - \lambda I| = 0$  is called the characteristic equation of  $A$  and its roots are called the characteristic roots (or latent roots or eigen values) of  $A$ .

**Remark.** It can be easily seen that the diagonal elements of  $A$  are of the first degree in  $\lambda$  and the off-diagonal elements are independent of  $\lambda$ .  $|A - \lambda I|$  is, therefore a polynomial of degree in  $\lambda$ , and the co-efficient of  $\lambda^n$  is  $(-1)^n$ . It follows that  $|A - \lambda I| = 0$  is of degree  $n$  in  $\lambda$ . By the fundamental theorem of algebra, every polynomial equation of degree  $n$  has exactly  $n$  roots. Therefore it follows that every square matrix of order  $n$  has exactly  $n$  characteristic roots.

**Example 1.** Find the characteristic roots of the matrix.

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 0 & 3 & 7 \\ 0 & 0 & 5 \end{pmatrix}$$

**Solution.**

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 & 4 \\ 0 & 3-\lambda & 7 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)(5-\lambda)$$

The roots of  $|A - \lambda I| = 0$  are, therefore, 1, 3, and 5.

Hence the characteristic roots of  $A$  are 1, 3, and 5.

**Example 2.** Show that the matrices  $A$  and  $A'$  have the same characteristic roots.

**Solution.** Let  $A = [a_{ij}]$  be an  $n \times n$  matrix. The characteristic equation of  $A$  is

$$|A - \lambda I| = 0 \tag{i}$$

Also the characteristic equation of  $A'$  is

$$|A' - \lambda I| = 0,$$

which is the same as

$$|(A - \lambda I)'| = 0 \tag{ii}$$

Since the value of a determinant remains unaltered by taking its transpose, therefore

$$|A - \lambda I| = |(A - \lambda I)'|$$

Consequently

$$|A - \lambda I| = 0, \Leftrightarrow |(A - \lambda I)^t| = 0 \quad (\text{iii})$$

i.e.,  $|A - \lambda I| = 0$  and  $|(A - \lambda I)^t| = 0$  are the same equation. Since  $|A - \lambda I| = 0$  is the characteristic equation of  $A$ , and  $|(A - \lambda I)^t| = 0$  is the characteristic equation of  $A^t$ , it follows that  $A$  and  $A^t$  have the same characteristic roots.

**Example 3.** Show that the characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

**Solution.** Let  $A = \begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ 0 & a_{22} & \dots & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}$  be an  $n \times n$  triangular matrix.

The characteristic equation of  $A$  is

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ 0 & a_{22} - \lambda & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$

$$\text{i.e., } (a_{11} - \lambda) (a_{22} - \lambda) \dots (a_{nn} - \lambda) = 0. \quad (\text{i})$$

The roots of (i) are given by

$\lambda = a_{11}, a_{22}, a_{33}, \dots$ , and  $a_{nn}$  which are just the diagonal elements of  $A$ .

Hence the characteristic roots of  $A$  are just the diagonal elements of  $A$ .

**Example 4.** Show that if  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the characteristic roots of  $A$ . then  $k\lambda_1, k\lambda_2, \dots, k\lambda_n$  are the characteristic roots of  $kA$ .

**Solution.** Let  $A = [a_{ij}]$  be an  $n$ -rowed square matrix. The characteristic equation of  $A$  is

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$

Let the roots of (i) be  $\lambda_1, \lambda_2, \dots, \lambda_n$ . to obtain the equation whose roots are  $k\lambda_1, k\lambda_2, \dots, k\lambda_n$ , we have to substitute  $\lambda^* = k\lambda$  in (i). The resulting equation in  $\lambda^*$  will have roots  $k\lambda_1, k\lambda_2, \dots, k\lambda_n$ .

By the transformation  $\lambda^* = k\lambda$ , equation (i) is transformed to

$$\begin{vmatrix} a_{11} - \frac{\lambda^*}{k} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \frac{\lambda^*}{k} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} - \frac{\lambda^*}{k} \end{vmatrix} = 0$$

$$\Leftrightarrow \left(\frac{1}{k}\right)^n \begin{vmatrix} ka_{11} - \lambda^* & ka_{12} & \dots & ka_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ ka_{n1} & ka_{n2} & \dots & ka_{nn} - \lambda^* \end{vmatrix} = 0$$

$$\Leftrightarrow \begin{vmatrix} ka_{11} - \lambda^* & ka_{12} & \dots & ka_{nn} - \lambda^* \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & ka_{22} - \lambda^* & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ ka_{n1} & ka_{n2} & \dots & ka_{nn} - \lambda^* \end{vmatrix} = 0$$

$$\Leftrightarrow |kA - \lambda^* I| = 0 \quad \dots(ii)$$

Since (ii) is the characteristic equation of  $kA$ , therefore it follows that the characteristics roots of (ii) are  $k$  times characteristics roots of (i), i.e., the characteristics roots of  $kA$  are  $k\lambda_1, k\lambda_2, \dots, k\lambda_n$ .

**Example 5.** If  $A$  and  $P$  be square matrices of the same type and if  $P$  be invertible, show that the matrices  $A$  and  $P^{-1}AP$  have the same characteristic roots.

**Solution.** The characteristics matrix of  $P^{-1}AP$  is  $P^{-1}AP - \lambda I$ .

$$\text{Now} \quad P^{-1}AP - \lambda I = P^{-1}(A - P(\lambda I)P^{-1})P,$$

$$= P^{-1}(A - \lambda I)P, \quad \dots(i)$$

$$\text{since } P(\lambda I)P^{-1} = (\lambda I)P P^{-1} = (\lambda I)I = \lambda I.$$

The characteristic equation of  $P^{-1}AP$  is  $|P^{-1}AP - \lambda I| = 0$ . Using (i) we find that this equation is the same is

$$|P^{-1}(A - \lambda I)P| = 0,$$

$$\begin{aligned}
&\Leftrightarrow |P^{-1}| |A - \lambda I| |P| = 0, \text{ since the determination of the product of two or more matrices} \\
&\hspace{15em} \text{is equal to the product of the determinants of the matrices.} \\
&\Leftrightarrow |P^{-1}| |P| |A - \lambda I| = 0, \text{ since } |P^{-1}|, |A - \lambda I|, |P| \text{ commute} \\
&\Leftrightarrow |P^{-1}P| |A - \lambda I| = 0, \text{ since the determinant of a product is equal to the product of} \\
&\hspace{15em} \text{determinants} \\
&\Leftrightarrow |I| |A - \lambda I| = 0, \text{ since } P^{-1}P = I \\
&\Leftrightarrow |A - \lambda I| = 0, \text{ since } |I| = 1
\end{aligned}$$

Since  $|P^{-1}AP - \lambda I| = 0 \Leftrightarrow |A - \lambda I| = 0$ , it follows that the characteristic roots of  $P^{-1}AP$  are the same as the characteristic roots of  $A$ .

### Exercise

- Find the characteristics roots of each of the following matrices:

$$(i) \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{pmatrix} \quad (ii) \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

- Show that the matrices.

$$\begin{pmatrix} o & g & f \\ g & o & h \\ f & h & o \end{pmatrix}, \begin{pmatrix} o & f & h \\ f & o & g \\ h & g & o \end{pmatrix}, \begin{pmatrix} o & h & g \\ h & o & f \\ g & f & o \end{pmatrix}$$

have the same characteristic equation.

- Show that the matrices

$$\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}, \begin{pmatrix} b & c & a \\ c & a & b \\ a & b & c \end{pmatrix}, \begin{pmatrix} c & a & b \\ a & b & c \\ b & c & a \end{pmatrix}$$

- Show that the characteristics roots of  $A^*$  are the conjugates of the characteristics roots of  $A$ .
- Show that 0 is a characteristics root of a matrix if and only if the matrix is singular.
- If  $A$  and  $B$  are  $n$ -rowed square matrices and if  $A$  be invertible, show that the matrices  $A^{-1}B$  and  $BA^{-1}$  have the same characteristic roots.

### 2. Cayley-Hamilton Theorem

We shall now establish the relation between a square matrix and its characteristics equation. The relation, which is known after it discovers Arthur Cayley and W.R. Hamilton is known as Cayley-Hamilton Theorem. It was established by Hamilton in 1953 for a special class of matrices and was stated (without proof) by Cayley in 1858.

**Theorem.** Let  $|A - \lambda I| = (-1)^n = \{\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_n\}$  be the characteristics polynomial of an  $n \times n$  matrix  $A$ . Then the matrix equation  $X^n + a_1X^{n-1} + a_2X^{n-2} + \dots + a_nI = O$  is satisfied by  $X = A$ .

**Proof.** The elements of  $A - \lambda I$  are at most of the first degree in  $\lambda$ . (In fact if we write the matrix in full, we find that the diagonal elements are of the first degree in  $\lambda$ , and all the off-diagonal elements

are independent of  $\lambda$ ). The elements of  $\text{adj}(A - \lambda I)$  are  $(n - 1) \times (n - 1)$  determinants, whose elements are at most of the first degree in  $\lambda$ . Therefore  $\text{adj}(A - \lambda I)$  is an  $n \times n$  matrix whose elements are at most of degree  $n - 1$  in  $\lambda$ . The matrix  $\text{adj}(A - \lambda I)$  can, therefore, be written as a matrix polynomial of degree  $n - 1$  in  $\lambda$ .

Let  $\text{adj}(A - \lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \dots + B_{n-1}$ , where  $B_0, B_1, \dots, B_{n-1}$  are  $n \times n$  matrices whose elements are functions of  $a_{ij}$ 's (the elements of  $A$ ).

Since  $(A - \lambda I) \text{adj}(A - \lambda I) = |A - \lambda I| I$ ,

Therefore  $(A - \lambda I) (B_0\lambda^{n-1} + B_1\lambda^{n-2} + \dots + B_{n-1}) = (-1)^n [\lambda^n + a_1\lambda^{n-1} + \dots + a_n] I_n$ .

Comparing the co-efficients of like powere of  $\lambda$  on both sides, we hve

$$\begin{aligned} -IB_0 &= (-1)^n I, \\ AB_0 - IB_1 &= (-1)^n a_1 I, \\ AB_1 - IB_2 &= (-1)^n a_2 I, \\ &\dots\dots\dots \\ AB_{n-1} &= (-1)^n a_n I. \end{aligned}$$

Premultiplying the above equations by  $A^n, A^{n-1} \dots I$ , respectively and adding the corresponding sides of the resulting equation, we have

$$O = (-1)^n [A^n + a_1 A^{n-1} + \dots + a_n I].$$

Thus  $A^n + a_1 A^{n-1} + \dots + a_n I = O$ , and this proves the theorem.

**Remark.** Cayley-Hamilton theorem is often expressed by saying that *every square matrix satisfies its characteristics equation.*

We shall now illustrate the Cayley-Hamilton theorem with the help of an example.

**Example 6.** Verify that the matrix  $A = \begin{pmatrix} 1 & 0 & 2 \\ 6 & 1 & 2 \\ 1 & -2 & 0 \end{pmatrix}$  satisfies its characteristics equation.

**Soluiton.** For the given matrix  $A$ ,

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 0 & 2 \\ 6 & 1-\lambda & 2 \\ 1 & -2 & -\lambda \end{vmatrix} \\ &= (1 - \lambda) [ \{- \lambda (1 - \lambda) + 4 \} + 2 \{- 12 - (1 - \lambda) \} ] \\ &= (1 - \lambda) (\lambda^2 - \lambda + 4) + 2 (\lambda - 13) \\ &= -\lambda^3 + 2\lambda^2 - 3\lambda - 22 \end{aligned}$$

The characteristics equation of  $A$  is therefore

$-\lambda^3 + 2\lambda^2 - 3\lambda - 22 = 0$   
or  $-\lambda^3 + 2\lambda^2 + 3\lambda + 22 = 0$  ...(i)

Now  $A^2 = \begin{pmatrix} 6 & 0 & 2 \\ 6 & 1 & 2 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 6 & 0 & 2 \\ 6 & 1 & 2 \\ 1 & -2 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 3 & -4 & 2 \\ 14 & -3 & 14 \\ -11 & -2 & -2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 3 & -4 & 2 \\ 14 & -3 & 14 \\ -11 & -2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 6 & 1 & 2 \\ 1 & -2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -19 & -8 & -2 \\ 10 & -31 & 22 \\ -25 & 2 & -26 \end{pmatrix}$$

Substituting  $A$  for  $\lambda$  in the left-hand side of (i) we have

$$A^3 - 2A^2 + 3A + 22I$$

$$= \begin{pmatrix} -19 & -8 & -2 \\ 10 & -31 & 22 \\ -25 & 2 & -26 \end{pmatrix} - 2 \begin{pmatrix} 3 & -4 & 2 \\ 14 & -3 & 14 \\ -11 & -2 & -2 \end{pmatrix} + 3 \begin{pmatrix} 1 & 0 & 2 \\ 6 & 1 & 2 \\ 1 & -2 & 0 \end{pmatrix} + 22 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Thus we have

$A^3 - A^2 + 3A + 22I = 0$ , showing that the matrix  $A$  satisfies its characteristic equation.

### 3. Application of Cayley-Hamilton Theorem to Compute Powers and Inverse of a given Square Matrix

Cayley-Hamilton theorem can be used to compute powers of a square matrix and invers of a non-singular square matrix.

Let the characteristic equation of an  $n$ -rowed square matrix  $A$  be

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_n = 0. \quad \dots(i)$$

By Cayley-Hamilton theorem the matrix  $A$  satisfies (i).

Therefore we have from (i),

$$A^m + a_1A^{m-1} + a_2A^{m-2} + \dots + a_nA^{m-n} = 0. \quad \dots(ii)$$

Substituting  $m = n + 1, n + 2, \dots, n + m$  in (ii) we have the relations

$$A^{n+1} + a_1A_n + a_2A^{n-1} + \dots + a_nA = 0. \quad \dots(iii)$$

$$A^{n+2} + a_1A^{n+1} + a_2A^n + \dots + a_nA^2 = 0.$$

.....

$$A^{n+k} + a_1A^{n+k-1} + a_2A^{n+k-2} + \dots + a_nA^k = 0.$$

By substituting the values of  $A, A_2, \dots, A^n$  in the first of the above equations, we get the value of  $A^{n+1}$ . Similarly by substituting the values  $A^2, A^3, \dots, A^{n+1}$  in the second of the above equations we can get the value of  $A^{n+2}$ , and so on.

If the matrix  $A$  is non-singular, then  $a_n \neq 0$ .



But multiplying (ii) throughout by  $\frac{1}{a_n}A^{-1}$  we have

$$\left(\frac{1}{a_n}\right)A^{n-1} + \left(\frac{a_1}{a_n}\right)A^{n-2} + \dots + A^{-1} = 0$$

or 
$$A^{-1} = \left(-\frac{1}{a_n}\right)A^{n-1} + \left(-\frac{a_1}{a_n}\right)A^{n-2} + \dots + \left(-\frac{a_{n-1}}{a_n}\right)I \quad \dots(iv)$$

By substituting the values of  $I, A, A^2, \dots, A^{n-1}$  in the right-hand side of (iv) and simplifying we can get the value of  $A^{-1}$ .

**Example 7.** Find the characteristic equation of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & -1 \end{pmatrix},$$

and hence compute its cube.

**Solution.** The characteristic equation of the matrix  $A$  is

$$? \begin{vmatrix} 1-l & 2 & 3 \\ 2 & 3-\lambda & 4 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0.$$

$$\Leftrightarrow (1-\lambda)(3-\lambda)(-1-\lambda) - \{2(-1-\lambda) - 4\} + 3(-3+\lambda) = 0$$

$$\Leftrightarrow \lambda^3 - 3\lambda^2 + 8\lambda = 0 \quad \dots(i)$$

By Cayley-Hamilton theorem by substituting  $A$  for  $\lambda$  in (i), we have

$$? A^3 - 3A^2 + 8A = 0,$$

so that 
$$A^3 = 3A^2 + 8A. \quad \dots(ii)$$

$$\text{Now } A^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & -1 \end{pmatrix}$$

$$? = \begin{pmatrix} 8 & 8 & 8 \\ 12 & 13 & 14 \\ 0 & 2 & 4 \end{pmatrix} \quad \dots(iii)$$

Substituting the values of  $A^2$  and  $A$  in (ii) we have

$$\begin{aligned} A^3 &= 3 \begin{pmatrix} 8 & 8 & 8 \\ 12 & 13 & 14 \\ 0 & 2 & 4 \end{pmatrix} + 8 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 32 & 40 & 48 \\ 52 & 63 & 74 \\ 8 & 6 & 4 \end{pmatrix}. \end{aligned}$$

**Example 8.** Find the characteristics equation of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

and hence compute its inverse.

**Solution.** The characteristic equation of the matrix  $A$  is

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 1-\lambda & 2 \\ 1 & 2 & -\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (1 - \lambda) \{-\lambda(1 - \lambda) - 4\} + 2(-1 + \lambda) = 0$$

$$\Leftrightarrow \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0. \quad \dots(i)$$

By Cayley-Hamilton theorem,  $A$  satisfies (i) so that we must have

$$A^3 - 2A^2 - 5A + 6I = 0. \quad \dots(ii)$$

Multiplying (ii) throughout by  $A^{-1}$  we have

$$A^2 - 2A - 5I + 6A^{-1} = 0,$$

so that

$$\begin{aligned} 6A^{-1} &= -A^2 + 2A + 5I \\ &= -\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}^2 + 2\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= -\begin{pmatrix} 3 & 4 & 2 \\ 2 & 5 & 2 \\ 1 & 2 & 6 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -4 & 2 \\ -2 & 2 & 2 \\ 1 & 2 & -1 \end{pmatrix} \end{aligned}$$

whence 
$$A^{-1} = \frac{1}{6} \begin{pmatrix} 4 & -4 & 2 \\ -2 & 2 & 2 \\ 1 & 2 & -1 \end{pmatrix},$$

$$= \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \end{pmatrix}$$

**Verification.** By actual multiplication it can be easily seen that  $AA^{-1} = I$ .

#### 4. Application of Cayley-Hamilton Theorem : Simplification of Matrix Polynomials

By using Cayley-Hamilton theorem of an  $n$ -rowed square matrix.  $A$ , a polynomial in  $A$  of degree greater than  $n$  can be reduced to a polynomial in  $A$  of degree less than  $n$ , i.e., a polynomial of degree greater than two in a  $2 \times 2$  matrix  $A$  can be reduced to a linear polynomial in  $A$ , a polynomial of degree greater than three in a  $3 \times 3$  matrix  $A$  can be reduced to a quadratic polynomial in  $A$ , and so on. The following example will illustrate the method.

**Example 9.** If  $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$ , express  $2A^5 - 3A^4 + A^2 - 4I$  as a linear polynomial in  $A$ .

**Solution.** The characteristic equation of  $A$  is

$$\begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0,$$

i.e.,  $(3 - \lambda)(2 - \lambda) + 1 = 0,$

i.e.,  $\lambda^2 - 5\lambda + 7 = 0.$

The given polynomial is  $f(A)$ , where

$$f(\lambda) = 2\lambda^5 - 3\lambda^4 + \lambda^2 - 4.$$

By division algorithm for polynomials we can write

$$f(\lambda) = (\lambda^2 - 5\lambda + 7)(2\lambda^3 + 7\lambda^2 + 21\lambda + 57) + 138\lambda - 403$$

$$\therefore f(A) = (A^2 - 5A + 7I)(2A^3 + 7A^2 + 21A + 57I) + (138A - 403I) \quad \dots(i)$$

Since every square matrix satisfies its characteristic equation, therefore

$$A^2 - 5A + 7I = 0.$$

Substituting from (i) in (i) it follows that  $f(A) = 138A - 403I$ .

#### Exercise

1. Verify that the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix} \text{ satisfies its characteristic equation.}$$

2. Obtain the characteristic equation of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 6 \end{pmatrix} \text{ and hence calculate its inverse.}$$

3. If  $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ , express

$$A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2 \text{ as a linear polynomial in } A.$$

4. Evaluate the matrix polynomial  $A^5 - 27A^3 + 65A^2$  as a  $3 \times 3$  matrix, where

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix}$$

5. If  $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

show that  $A^3 - A = A^2 - I$ .

Deduce that for every integer  $n \geq 3$ ,  $A^n - A^{n-2} = A^2 - I$ . Hence otherwise determine  $A^{50}$

S.R.S. 1

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Exercise

- Show that the set  $C_2 = \{(x_1, x_2) : x_1 \in C, x_2 \in C\}$  is a vector space over  $C$  with respect to co-ordinate wise addition and scalar multiplication.
- Show that the set of all  $2 \times 2$  matrices over  $C$  is a vector space over  $C$  with respect to matrix addition and multiplication of a matrix by a scalar.
- Let  $V = \{(a_1, a_2, a_3, a_4) : a_1, a_2, a_3, a_4 \text{ are integers}\}$   
Is  $V$  a vector space over  $R$  with respect to co-ordinatewise addition and scalar multiplication ?  
Justify your answer.
- Let  $V = \{(x_1, x_2, x_3) : x_1, x_2, x_3, \text{ are complex numbers, and } x_1, x_2, = 0\}$   
Is  $V$  a vector space of over  $C$  with respect to co-ordinatewise addition and scalar multiplication ?  
Justify your answer.
- Show that the set of all matrices of the form  $\begin{pmatrix} x & o \\ o & y \end{pmatrix}$ , where  $y \in C$  is a vector space over  $C$  with respect to matrix addition and multiplication of a matrix by a scalar.
- Show that the set of all matrices of the form  $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ , where  $a, b \in C$  is a vector space over  $C$ . With respect to matrix addition and multiplication of a matrix by a scalar.

**S.R.S. 2**

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**Exercise**

1. For each of the following matrices  $A$ , verify that  $A^t = A$ :

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & -4 \\ -1 & -4 & 3 \end{pmatrix}, \begin{pmatrix} -1 & 6 & -7 \\ 6 & 3 & 8 \\ -7 & 8 & -5 \end{pmatrix}, \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$$

2. For each of the following matrices  $A$ , verify that  $A^t = -A$ :

$$\begin{pmatrix} 0 & i & 1 \\ -i & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 1+i \\ -3 & 0 & -4 \\ -1-i & 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -3 & 2i \\ 3 & 0 & 4 \\ -2i & -4 & 0 \end{pmatrix}$$

3. If  $A = \frac{1}{3} \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}$ , verify that  $A A^t = A^t A = A = I_3$ .

4. If  $A$  be any square matrix, verify that the matrix  $A + A^t$  is symmetric and the matrix  $A - A^t$  is skew-symmetric.

5. If  $A$  be any square matrix, prove that  $A + A^0, AA^0, A^0A$  are all hermitian and  $A - A^0$  is skew-hermitian.

6. Express the matrix  $\begin{pmatrix} 3 & 1 & -7 \\ 2 & 4 & 8 \\ 6 & -1 & 2 \end{pmatrix}$  as  $X + Y$  where  $X$  is symmetric and  $Y$  is skew-symmetric.

7. Find remain matrices  $A$  and  $B$  such that

$$A + iB = \begin{pmatrix} 1 & 1+i & 3 \\ 4 & 2 & 1-i \\ -3 & 2+3i & i \end{pmatrix}$$

**S.R.S. 3(a)**

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**Exercise**

1. Apply the elementary operation  $R_2 \leftrightarrow R_3$  to the matrix

$$\begin{pmatrix} 1 & -1 & 3 \\ 4 & 2 & -6 \\ 5 & 8 & 9 \end{pmatrix}$$

2. Apply the elementary operation  $C_2 \rightarrow 2C_2$  to the matrix

$$\begin{pmatrix} -1 & 3 & 7 & 6 \\ 5 & -1 & 4 & -2 \end{pmatrix}$$

3. Write down the elementary matrix obtained by applying  $R_3 \rightarrow R_3 - 4R_1$  to  $I_3$ .

4. Reduce the matrix

$$\begin{pmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 2 & 0 \\ 2 & 1 & -1 & 5 \end{pmatrix}$$

to triangular form by applying E-row operations

5. Reduce the matrix

$$\begin{pmatrix} 1 & -1 & -1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{pmatrix}$$

to I3 by E-row operations

6. Verify that the E-row operation  $R_1 \rightarrow R_1 - R_3$  on the matrix  $AB$  is equivalent to the same E-row operation on  $A$ , where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -3 & 1 & 4 \\ 0 & 1 & 3 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 & -5 \\ 1 & -2 & 6 \\ 3 & 1 & 1 \end{pmatrix}.$$

**S.R.S. 3(b)**

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**Exercise**

1. Find the inverse of the matrix

$$\begin{pmatrix} 3 & -15 & 5 \\ -1 & 6 & -2 \\ 1 & -5 & 2 \end{pmatrix}$$

by calculating its adjoint.

2. Show that the matrix

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{pmatrix}$$

possesses an inverse.

3. Given that  $A = \begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ , calculate  $A^2$  and  $A^{-1}$

4. If  $A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -9 & 5 \end{pmatrix}$ ,  $B = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ , verify that  $(AB)^{-1} = B^{-1}A^{-1}$ .

5. By applying elementary row operations find the inverse of the matrix

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & -3 \\ 2 & 1 & -1 \end{pmatrix}.$$



**S.R.S. 4**

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**Exercise**

1. Find the rank of the matrix

$$\begin{pmatrix} 3 & -1 & 2 & -6 \\ 6 & -2 & 4 & -9 \end{pmatrix}.$$

2. Find the rank of the matrix.

$$\begin{pmatrix} -1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

3. Find the rank of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{pmatrix}.$$

4. Reduce the matrix

$$\begin{pmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

to triangular form and hence find rank.

5. Find the rank of the matrix

$$\begin{pmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{pmatrix}.$$

**S.R.S. 5**

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**Exercise**

1. Does the following system of equations possess a unique common solution?

$$\begin{aligned} x + 2y + 3z &= 6, \\ 2x + 4y + z &= 7, \\ 3x + 2y + 9z &= 10. \end{aligned}$$

It so, find it.

2. Show that  $x = y = z = 0$  is the only common solution of the following system of equations:

$$\begin{aligned} 2x + 3y + 4z &= 0, \\ x - 2y - 3z &= 0, \\ 3x + y - 8z &= 0. \end{aligned}$$

3. Solve the system of equations :

$$\begin{aligned} x + y + z &= 3, \\ 3x - 5y + 2z &= 8, \\ 5x - 3y + 4z &= 14. \end{aligned}$$

4. Solve the following system of equations :

$$\begin{aligned} x - 3y + 2z &= 0, \\ 7x - 21y + 14z &= 0, \\ -3x + 9y - 6z &= 0. \end{aligned}$$

5. Which of the following system of equations are consistent ?

(a)

$$\begin{aligned} x - 4y + 7z &= 8, \\ 3x + 8y - 2z &= 6, \\ 7x - 8y + 26z &= 31. \end{aligned}$$

(b)

$$\begin{aligned} x - y + 2z &= 4, \\ 3x + y + 4z &= 6, \\ x + y + z &= 1. \end{aligned}$$

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Exercise

- Find the characteristics roots of each of the following matrices:

$$(i) \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{pmatrix} \quad (ii) \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

- Show that the matrices.

$$\begin{pmatrix} o & g & f \\ g & o & h \\ f & h & o \end{pmatrix}, \begin{pmatrix} o & f & h \\ f & o & g \\ h & g & o \end{pmatrix}, \begin{pmatrix} o & h & g \\ h & o & f \\ g & f & o \end{pmatrix}$$

have the same characteristics equation.

- Show that the matrices

$$\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}, \begin{pmatrix} b & c & a \\ c & a & b \\ a & b & c \end{pmatrix}, \begin{pmatrix} c & a & b \\ a & b & c \\ b & c & a \end{pmatrix}$$

- Show that the characteristics roots of  $A^*$  are the conjugates of the characteristics roots of  $A$ .
- Show that 0 is a characteristics root of a matrix if and only if the matrix is singular.
- If  $A$  and  $B$  are  $n$ -rowed square matrices and if  $A$  be invertible, show that the matrices  $A^{-1}B$  and  $B A^{-1}$  have the same characteristics roots.

**S.R.S. 6(b)**

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**Academic Session 2012–2013**

**Exercise**

1. Verify that the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{pmatrix} \text{ satisfies its characteristic equation.}$$

2. Obtain the characteristic equation of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 6 \end{pmatrix} \text{ and hence calculate its inverse.}$$

3. If  $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ , express

$$A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2 \text{ as a linear polynomial in } A.$$

4. Evaluate the matrix polynomial  $A^5 - 27A^3 + 65A^2$  as a  $3 \times 3$  matrix, where

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{pmatrix}$$

5. If  $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

show that  $A^3 - A = A^2 - I$ .

Deduce that for every integer  $n \leq 3$ ,  $A^n - A^{n-2} = A^2 - I$ . Hence otherwise determine  $A^{50}$ .